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Control Approach in Cloaking Problems for 2-D Model of Sound Scattering

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Abstract. We consider control problems for 2-D Helmholtz equation in a bounded domain with partially coated boundary. These problems are associated with acoustic cloaking. Dirichlet boundary condition is given on one part of the boundary and the impedance boundary condition is given on another part of the boundary. The role of control in control problem under study is played by surface impedance. Solvability of control problem is proved and optimality system is derived.

Introduction. Statement and solvability of the boundary value problem

In this paper we consider time-harmonic acoustic waves in an infinitely long cylindrical waveguide with bounded cross section Ω. It is assumed that domain Ω is subset of R^2 and the boundary Γ of the domain Ω is Lipshitz and consists of two parts Γ_D and Γ_I which are disjoint, relatively open subsets on Γ. We also assume that boundary conditions of Dirichlet and impedance type are specified on uncoated part Γ_D and coated part Γ_I of the boundary Γ, respectively. Let ν denote the unit outward normal vector defined almost everywhere on Γ. It is well known that the two-dimensional direct scattering problem in this situation is described by the 2-D Helmholtz equation

$$\Delta u+k^2u=0 \text{ in } \Omega \quad (1)$$

with mixed boundary conditions

$$u=0 \text{ on } \Gamma_D,\ \partial u/\partial\nu + ik\lambda u = g \text{ on } \Gamma_I. \quad (2)$$

Here λ is a surface impedance on the part Γ_I of the boundary Γ, k is a positive wave number, g is a function defined on Γ_I. Boundary condition on Γ_I implies that the normal component of vibrational velocity is proportional to acoustic pressure. We shall refer to problem defined by Eq. 1 and Eq. 2 as Problem 1.

The direct scattering problem defined by Eq. 1 and Eq. 2 was formulated and studied in [1] where the unique solvability of the solution is proved in the case when λ = const > 0. A number of papers is devoted to study of coefficient inverse problems of determining the surface impedance λ (or finding surface conductivity) from the far field data (see, e.g., [2, 3] and references therein). In [4, 5] boundary control problems for the 3-D Maxwell system considered in a bounded domain under impedance boundary condition were formulated and studied using nonlinear optimization techniques. We also mention papers [6, 7, 8] devoted to application of optimization methods for solving related problems of technical gas dynamics.

The goal of this paper is analysis of control problems connecting with solving cloaking problems for electromagnetic or acoustic fields. Beginning with pioneering paper by J. Pendry et al. [9] the large number of publications was devoted to solving of problems of constructing cloaking shells (see, e.g., [10, 11] and references therein). In cited papers cloaking effect is achieved due choice of parameters of medium filling cloaking shell by solving respective inverse problems for Maxwell equations or Helmholtz equation with variable coefficients. We emphasize that technical realization of solutions obtained in these papers is connected with substantial technical difficulties.

There are several approaches of overcoming these difficulties. The first approach consists of approximation of solutions of “exact” cloaking problem by approximate solutions which admit simple technical realization. Alternative approach consists of replacing the exact cloaking problem by approximate cloaking problem of constructing “approximate” cloaking shell. This approach is based on introducing the cost functional under minimization which adequately corresponds to inverse problem of constructing approximate cloaking shell. Just this idea is used in this paper. (The same goal can be realized by shape optimization of the boundary; we refer the reader to [12] for the related results on shape sensitivity analysis for the coupled models.) Moreover, unlike cited papers the cloaking effect in the paper is achieved due choice of surface impedance λ entering into the second boundary condition in Eq. 2. The details of this approach can be found in [13, 14, 15]. For realization of this purpose we formulate and study control problem for model Eq. 1.

In this section we study briefly solvability and uniqueness of solutions of direct boundary value problem. We shall use Sobolev space $H^1(\Omega)$ consisting of complex or real valued scalar functions defined in domain Ω and the trace spaces $H^{1/2}(\partial\Omega)$ and $H^{1/2}(\Gamma_0)$ where Γ_0 is a part of $\partial\Omega$. We also shall use subspace X of space $H^1(\Omega)$ which consists of functions v from $H^1(\Omega)$ which satisfy first boundary condition from Eq. 2. The norms in spaces $H^1(\Omega)$, $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$ are denoted by $\| \|_{1,\Omega}$, $\| \|_{1/2,\Gamma 0}$ and $\| \|_{-1/2,\Gamma 0}$. The inner products and norms in $L^2(Q)$ are denoted by $(\, , \,)_Q$ and $\| \|_Q$. If $Q = \Omega$ then we set $\| \|_\Omega = \| \|$, $(\, , \,)_\Omega = (\, , \,)$. The inner products and norms in $L^2(\Gamma_0)$ are denoted by $(\, , \,)_{\Gamma 0}$ and $\| \|_{\Gamma 0}$. See more details in [16].

Now we study briefly solvability and uniqueness of solutions of direct boundary value problem defined by Eq. 1 and Eq. 2. To this purpose we multiply Eq. 1 by a function v* where v is element of X (we denote complex conjugate of v as v* for any function or functional), then integrate over Ω and apply Green’s formula. We obtain

$$a_0(u, v) + ik(\lambda u, v)_{\Gamma I} = (g, v)_{\Gamma I}. \tag{3}$$

where

$$a_0(u, v) = (\text{grad } u, \text{grad } v) - k^2(u, v) , (u, v) = \int_\Omega uv^*dx, (g, v)_{\Gamma I} = \int_{\Gamma I} gv^*d\sigma. \tag{4}$$

We call a solution u of problem Eq. 3 a weak solution of Problem 1.

Using the theory developed in [16] we can prove the following theorem.

Theorem 1. Let λ be an element of $L^\infty(\Gamma_I)$ and $\lambda(x) \geq \lambda_0$; $\lambda_0 > 0$. Then for any function g from $L^2(\Gamma_I)$ problem Eq. 3 has a unique solution u which is an element of X.

Statement and solvability of the control problem

The control problem under study consists of minimization of certain cost functional depending on the state u and an unknown function (control) satisfying the state Eq. 1 and Eq. 2. As the cost functional we choose one of the following:

$$I_1(u) = \int_Q |u - u_d|^2dx, \; I_2(u) = \int_{\Gamma r} |u - u_d|^2d\sigma. \tag{5}$$

Here u_d an element of $L^2(Q)$ (or u_d an element of $L^2(\Gamma_r)$) which models the acoustic field measured in some subdomain Q of domain Ω or on the boundary Γ_r of the disc B_r of the radius *r* such that B_r is subdomain of Ω.

Now we are able to state and study our control problem. We shall assume that control λ is changed over certain set K. More precisely, it is assumed that the following conditions are satisfied:

(j) Γ is an element of $C^{0,1}$, Γ_I is an element of $C^{1,1}$; $\alpha_0 > 0$; K is subset of $H^s_{\lambda 0}(\Gamma_I)$ which contains functions λ which are satisfy $\lambda(x) \geq \lambda_0$; K is non-empty convex closed set where $s > 1/2$, $\lambda_0 > 0$.

Rewrite the weak formulation of Problem 1 in the form of the operator equation G(u, λ, g) = 0. We consider the following constrained minimization problem:

$$J(u, \lambda) = (\alpha_0/2)I(u) + (\alpha_1/2)\|\lambda\|^2_{s,\Gamma I} \rightarrow \inf, G(u, \lambda, g) = 0, (u, \lambda) \text{ is an element of } X \times K. \quad (6)$$

Following theorems can be proved.

Theorem 2. Let under assumptions (j) I: X → R be a weakly lower semicontinuous functional and Z_{ad} be a non-empty set. Let further $\alpha_1 \geq 0$ and K be bounded set, or $\alpha_1 > 0$ and functional I be bounded from below. Then problem Eq. 6 has at least one solution (u, λ) which belongs to X × K.

Theorem 3. Let under assumptions (j), $\alpha_1 > 0$ or $\alpha_1 \geq 0$ and K be bounded set. Then control problem Eq. 6 has at least one solution (u, λ) which is an element of X × K for $I = I_m$, m = 1, 2.

The optimality system

The following stage of analysis of control problem Eq. 6 consists of derivation of the optimality system which describes the first–order necessary conditions of extremum for our control problem. For this purpose we make use of the approach developed in [17]. It is based on the derivation and analysis of an optimality system describing the first-order necessary conditions for an extremum in problem Eq. 6. However, since problem Eq. 6 is stated for complex valued functions, it has to be preliminarily decomplexified. As a result it is reduced to control problem considered in the class of real-valued functions. Then based on [17] the optimality system can be derived for the latter "real" control problem. Finally this "real" optimality system is transformed to "complex" optimality system corresponding to initial problem Eq. 6. On this way the following theorem can be proved.

Theorem 4. Let under assumptions (j) an element (u_1, λ_1) from X × K be a solution of problem Eq. 6 and let I(u) be continuously Frechet differentiable functional with respect to the state u in a point u_1. Then there exists a unique non-zero Lagrange multiplier p which is an element of X that satisfies the Euler-Lagrange equation

$$a_0(v, p) + ik(\lambda_1 v, p)_{\Gamma I} = -(\alpha_0/2)\,(\langle I'_u(u_1), v\rangle)^* \text{ for all } v \text{ in } X, \quad (7)$$

and following variational inequality holds

$$\alpha_1(\lambda_1, \lambda - \lambda_1)_{s,\Gamma I} + k\mathrm{Re}[i((\lambda - \lambda_1)u_1, p)_{\Gamma I}] \geq 0 \text{ for all } \lambda \text{ in } K. \quad (8)$$

The direct problem Eq. 3, identity Eq. 7 which has the sense of an adjoint problem for adjoint state and variational inequality Eq. 8 constitute the optimality system for control problems under study. It describes necessary conditions of an extremum for control problem Eq. 6.

The numerical algorithm

The optimality system plays an important role in investigating properties of solutions to the control problem. On the basis of the analysis of optimality systems, sufficient conditions for the initial data, which provide the uniqueness and stability of the solutions to individual extremum problems, can be formulated. Besides, efficient numerical algorithms can be developed for solving problem Eq. 6. (In the case of the 2-D cloaking problem, some of these algorithms were considered in [13]).

Optimality system derived above can be used to design effective numerical algorithms for solving control problem Eq. 6 under study. Simplest numerical algorithm can be obtained by applying simple iteration method for solving the optimality system. The n-th iteration of this algorithm consists of finding values u^n, p^n and λ^{n+1} by sequentially solving following problems:

$$a_0(u^n, v) + ik(\lambda^n u^n, v)_{\Gamma I} = (g, v)_{\Gamma I} \text{ for all } v \text{ in } X, \tag{9}$$

$$a_0(v, p^n) + ik(\lambda^n v, p^n)_{\Gamma I} = -(\alpha_0/2)\,(\langle I'_u(u^n), v\rangle)^* \text{ for all } v \text{ in } X, \tag{10}$$

$$\alpha_1(\lambda^{n+1,}\, \lambda - \lambda^{n+1})_{s,\,\Gamma I} + k\mathrm{Re}[i((\lambda - \lambda^{n+1})u^n, p^n)_{\Gamma I}] \geq 0 \text{ for all } \lambda \text{ in } K. \tag{11}$$

Direct and adjoint problems can be solved by finite element method. Authors plan study numerical algorithms for solving control problems and analyze results of numerical experiments in forthcoming papers.

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