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VISCOMETRIC FLOW OF ELASTOPLASTIC MATERIAL HEATED BY WALL FRICTION

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Abstract: A mathematical model of large deformations is used to solve a coupled boundary-value problem about the deformation of an elastoviscoplastic material in a cylindrical viscometer with account for its heating due to wall friction. The deformation of a material enclosed between rigid surfaces due to the rotation of an inner cylindrical surface at a variable velocity is investigated. It is taken into account that a yield point depends on temperature. The motion of elastoplastic boundaries is described. Stresses, strains, and temperature in a thermoelastic deformation region and in a flow region both during the development of the flow and during its deceleration, including stopping, unloading, and cooling, are calculated. Residual stresses and deformations are determined.

Keywords: elasticity, viscosity, plasticity, viscometric flow, large deformations, thermoplasticity.

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Results of viscometric experiments can be used to measure the viscosity of structural materials. These results are usually processed using the exact solution of the corresponding model boundary-value problem. In the mechanics of viscous and viscoplastic media, such solutions obtained within the Shvedov—Bingham rigid-plastic model are classical [1–4], and rather universal methods for calculating viscoplastic flows are developed [5–7]. The use of structural materials or elastic fluids in the cases where the elastic properties of materials cannot be neglected significantly complicates the problems of viscometric flows. In such cases, deformations in stagnant zones and moving cores are predominantly reversible and boundary-value problems should be formulated in displacements, while this problem in flow regions is solved in displacement velocities. Reversible deformation regions and flow regions are separated by an unknown moving boundary on which the displacement continuity condition should be fulfilled. The calculation of displacement components in flow regions is a rather complicated task [8], so few solutions to elastoplastic problems in the theory of flow have been obtained [9, 10]. Note that, in thecase of elastoviscoplastic media, the condition of equality of the velocities and stress components on the elastoplastic boundary is also insufficient, which can lead to erroneous solutions [11].

As irreversible deformations are large in flow regions, the problems of viscometric flow should be considered using the model of large elastoplastic deformations. There are many similar approaches to modeling elastoplastic properties [12–17]. This work is carried out using a model in which, in accordance with the laws of nonequilibrium thermodynamics, reversible and irreversible deformations considered as thermodynamic parameters of state are determined from differential transfer equations [18–21].

With the help of this approach, solutions to some theoretical problems (including exact analytical ones) have been obtained [21–25].

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Next, we generalize the solution obtained in [25] and in which a viscoplastic flow was considered with account for slippage on the walls of a fluid meter. Assuming that the presence of sliding friction heats up the deformable material, we consider the related problem of heat production and irreversible deformations due to viscometric deformation and wall friction. The problem is solved using a mathematical model of large elastoplastic deformations, constructed in [18–20], described in detail in [21], and generalized for a nonisothermal case [26] and a case where the viscous properties of the material during its plastic flow are accounted for [27]. Within the framework of such a mathematical model, the solutions of coupled thermomechanical problems were previously obtained for the cases of rectilinear motions of elastoviscoplastic materials [28–31].

1. BASIC RELATIONS OF THE MODEL

The reversible (thermoelastic) component m and irreversible component p of total deformations are determined by differential equations of their variation (transfer). In Euler variables, such relations have the form

$$\frac{Dp}{Dt} = \frac{dp}{dt} - x\cdot p + p\cdot x = \gamma - p\cdot\gamma - \gamma\cdot p,$$
$$\frac{Dm}{Dt} = \varepsilon - \gamma - \frac{1}{2}\left((\varepsilon-\gamma+z)\cdot m + m\cdot(\varepsilon-\gamma-z)\right), \tag{1.1}$$

where

$$x = -x^{\mathrm t} = w + z,\quad \varepsilon = \frac{1}{2}(\nabla\boldsymbol v + \nabla^{\mathrm t}\boldsymbol v),\quad w = \frac{1}{2}(\nabla\boldsymbol v - \nabla^{\mathrm t}\boldsymbol v),\quad \boldsymbol v = \frac{\partial\boldsymbol u}{\partial t} + \boldsymbol v\nabla\boldsymbol u,$$
$$m = e + \alpha T_0\theta I,\qquad \theta = T_0^{-1}(T - T_0),$$
$$z = -z^{\mathrm t} = A^{-1}(B^2(\varepsilon\cdot m - m\cdot\varepsilon) + B(\varepsilon\cdot m^2 - m^2\cdot\varepsilon) + m\cdot\varepsilon\cdot m^2 - m^2\cdot\varepsilon\cdot m), \tag{1.2}$$
$$A = 8 - 8J_1 + 3J_1^2 - J_2 - J_1^3/3 + J_3/3,\qquad B = 2 - J_1,$$
$$J_1 = I\cdot m,\qquad J_2 = I\cdot m^2,\qquad J_3 = I\cdot m^3,$$

$\boldsymbol u$, $\boldsymbol v$ are the displacement and velocity vectors; I is the second-rank unit tensor; T and T_0 are the current temperature and the free temperature (room temperature), respectively; α is the linear expansion coefficient; D/Dt is the objective time derivative. Temperature T (or the entropy distribution density S) and reversible (thermoelastic) m and irreversible p deformations are taken as the thermodynamic parameters of the state of a body during its deformation. According to the first equation (1.1), the deformation processes in which the irreversible deformations p remain unchanged are determined by the vanishing of the source of irreversible deformations γ. In this case, $Dp/Dt = 0$ and the irreversible strain tensor components p change in the same way as during rigid the displacement of the body. The objective derivative introduced in relation (1.1) coincides with the Yaumann derivative in the case of zero nonlinear addition: $z(\varepsilon, m) = 0$.

For the Almansi total strain tensor d, Eqs. (1.1) and (1.2) yield [21]

$$d = m + p - m\cdot m/2 - m\cdot p - p\cdot m + m\cdot p\cdot m. \tag{1.3}$$

The free energy $\Phi(m,T) = E(d,S) - TS$ ($E(d,S)$ is the internal energy distribution density) is used as a thermodynamic potential. A hypothesis that significantly simplifies the mathematical model and assumes that the free energy does not depend on irreversible deformations is taken, and it is believed that the conservative part of the deformation process is specified by the elastic potential $W(m,\theta) = \rho_0\Phi(m,T)$ (ρ_0 is the material density in free state). As the material is assumed to be mechanically incompressible, the Murnaghan formula and the entropy balance equation are derived from the law of conservation of energy:

$$\sigma = -pI + \frac{1}{1+3\alpha T_0\theta}\frac{\partial W}{\partial m}\cdot(I - m); \tag{1.4}$$

$$\frac{\partial(\rho S)}{\partial t} = -\operatorname{div}\boldsymbol J - T^{-2}\boldsymbol q\cdot\nabla T + T^{-1}\sigma\cdot\gamma, \tag{1.5}$$

Here p is the unknown hydrostatic pressure; $\boldsymbol q$ and $\boldsymbol J$ are the heat flux and entropy vectors:

$$\boldsymbol J = \rho S\boldsymbol v - T^{-1}\boldsymbol q,\qquad S = -T_0\rho^{-1}\frac{\partial W}{\partial\theta}.$$

For the considered case of an isotropic medium, the elastic potential $W(J_1, J_2, \theta)$ is expanded into a Taylor series with respect to the free state of the deformed material at room temperature T_0:

$$W(J_1, J_2, \theta) = -2\mu J_1 - \mu J_2 + \varkappa J_1^2 + (\varkappa - \mu) J_1 J_2 - \chi J_1^3 + \nu_1 J_1 \theta + \\ + \nu_2 \theta^2 - \nu_3 J_1 \theta^2 - \nu_4 J_1^2 \theta - \nu_5 J_2 \theta - \nu_6 \theta^3 + \ldots, \qquad (1.6)$$

$$J_1 = I \cdot c, \qquad J_2 = I \cdot c^2, \qquad c = m - 0{,}5m^2.$$

Here μ denotes the shear modulus, $\varkappa$ and χ are the higher-order elastic moduli, and ν_k $(k = 1, 2, \ldots, 6)$ denotes the thermomechanical constants.

Expression (1.6) is substituted into Eq. (1.5), and the heat equation is written as

$$(1 + \beta_1 \theta + \beta_2 J_1) \frac{\partial \theta}{\partial t} + \beta_3 (\varepsilon - \gamma) \cdot c = \lambda \Delta \theta - \frac{1}{2\nu_2} \sigma \cdot \gamma,$$
$$\beta_1 = \frac{(1 - 3\alpha T_0)\nu_2 - 3\nu_6}{\nu_2}, \qquad \beta_2 = -\frac{\nu_3}{\nu_2}, \qquad \beta_3 = -\frac{\nu_1 + \nu_5}{\nu_2} \qquad (1.7)$$

(λ is the thermal diffusivity). In the case where deformation precedes a viscoplastic flow and where there is unloading in (1.7), we set $\gamma = 0$. Thus, the production of irreversible deformations at these stages of the deformation process in the form of creep strains is neglected. In the flow region, $\gamma = \varepsilon^p$, i.e., the source term in the transfer equation (1.1) coincides with the plastic strain rate tensor.

The loading surface is specified by Tresca's yield condition with account for the viscous resistance to plastic flow [32]:

$$f(\sigma_i, \varepsilon_k^p, k) = \max |\sigma_i - \sigma_j| - 2k - 2\eta \max |\varepsilon_k^p|, \qquad (1.8)$$

Here σ_i and ε_k^p are the main values of the stress tensors and the plastic strain rates, k is the yield strength, and η is the viscosity.

The dependence between the yield strength and the temperature is written as [28–31]

$$k = k_0(1 - \theta^2/\theta_m^2), \qquad \theta_m = (T_m - T_0)T_0^{-1}, \qquad (1.9)$$

where T_m is the melting point of the deformed material and k_0 is the material yield strength at room temperature. The irreversible (plastic) deformation rates are related with stresses by the associated law of plastic flow

$$\varepsilon^p = \gamma = \varphi \frac{\partial f(\sigma, \gamma, k)}{\partial \sigma}, \qquad \varphi > 0. \qquad (1.10)$$

As Eqs. (1.7)–(1.10) are supplemented by the equilibrium equations $\nabla \sigma = 0$, a closed system of equations of quasistationary elastoviscoplastic deformation is obtained.

2. FORMULATION OF THE PROBLEM. INITIAL REVERSIBLE DEFORMATION

Let an elastoviscoplastic material fill an annular gap between rigid cylindrical surfaces $r = r_0$ and $r = R$ $(R > r_0)$. The inner cylinder rotates around its axis with a given variable angular velocity $\omega_{r_0}(t)$, while the outer cylinder remains steady.

The trajectories of the medium points are concentric circles, so, according to Eqs. (1.1)–(1.3) and the incompressibility condition, the kinematics of the medium in the cylindrical coordinate system (r, φ, z) is determined by the following relations

$$u_r = r(1 - \cos \psi(r, t)), \qquad u_\varphi = r \sin \psi(r, t), \qquad u_z = 0,$$
$$d_{rr} = -\frac{1}{2} r^2 \Big(\frac{\partial \psi}{\partial r}\Big)^2 = -2g^2, \qquad d_{r\varphi} = \frac{1}{2} r \psi_{,r} = g, \qquad \psi_{,r} = \frac{\partial \psi}{\partial r},$$
$$v_\varphi = r\omega = r\psi_{,rt}, \qquad \varepsilon_{r\varphi} = \frac{\partial d_{r\varphi}}{\partial t} = \frac{1}{2}(v_{\varphi,r} - r^{-1} v_r) = \frac{1}{2} r \psi_{rt}, \qquad (2.1)$$
$$w_{r\varphi} = -\psi_{,t} - \frac{r}{2} \psi_{,rt}, \qquad x_{r\varphi} = -\psi_{,t} + \frac{2m_{r\varphi}(1 - m_{\varphi\varphi})}{m_{rr} + m_{\varphi\varphi} - 2},$$

where $\psi = \psi(r, t)$ is the value of the central angle of torsion of the medium points and $\omega(r, t)$ is their angular velocity.

The deformation of the material with increasing, constant, and decreasing rotation velocities of the inner cylinder are considered. Therefore, the role of the boundary conditions of the problem is played the relations

$$\psi(R,t)=0, \qquad \omega(R,t)=0,$$

$$w_{r_0}(t)=\begin{cases} at, & 0\leqslant t\leqslant t_1,\\ at_1, & t_1\leqslant t\leqslant t_2,\\ at_1-b(t-t_2), & t_2\leqslant t\leqslant t_5,\end{cases} \tag{2.2}$$

$$\psi_{r_0}=\begin{cases} at^2/2, & 0\leqslant t\leqslant t_1,\\ at_1t-at_1^2/2, & t_1\leqslant t\leqslant t_2,\\ at_1t-at_1^2/2-b(t^2-t_2^2)^2/2, & t_2\leqslant t\leqslant t_5.\end{cases}$$

It is assumed that, until the beginning of the deformation process $t=0$, there are no deformations in the cylindrical layer, the temperature is equal to room temperature T_0, and the initial compression is uniform: $\sigma_{rr}(r,0)=\sigma_{\varphi\varphi}(r,0)=\sigma_{zz}(r,0)=\sigma_0=\text{const}$. It is also assumed that initially the material is deformed reversibly and contacts rigid walls in accordance with the law of dry friction:

$$|\sigma_{r\varphi}|\leqslant\delta\,|\sigma_{rr}|, \quad \boldsymbol{u}=\boldsymbol{0}, \quad \omega=\boldsymbol{0} \quad \text{for} \quad r=r_0, \quad r=R \tag{2.3}$$

(δ is the static friction coefficient).

According to Eqs. (1.4) and (1.6), the stress components under elastic deformation are determined by the relations

$$\sigma_{rr}=\sigma_{zz}=-(p+2\mu)-2(\varkappa+\mu)g^2=-\Sigma, \quad \sigma_{\varphi\varphi}=-\Sigma+4\mu g^2, \quad \sigma_{r\varphi}=2\mu g. \tag{2.4}$$

In system (2.4), reversible deformations are assumed to be so small that their third order can be neglected. For the results obtained, such an assumption is not essential, but it allows one to significantly simplify the calculations.

Integrating the equilibrium equations (quasistatic case)

$$\sigma_{rr,r}+r^{-1}(\sigma_{rr}-\sigma_{\varphi\varphi})=0, \qquad \sigma_{r\varphi,r}+2r^{-1}\sigma_{r\varphi}=0, \tag{2.5}$$

and accounting for Eqs. (2.1) and (2.4), we obtain a solution valid in a time interval where only elastic deformation of the material occurs:

$$\sigma_{r\varphi}=\frac{f(t)}{r^2}, \qquad \sigma_{rr}=\sigma_{zz}=\frac{f^2}{4\mu}\Big(\frac{1}{r_0^4}-\frac{1}{r^4}\Big)+\sigma_0,$$

$$\sigma_{\varphi\varphi}=\frac{f^2}{4\mu}\Big(\frac{1}{r_0^4}-\frac{3}{r^4}\Big)+\sigma_0, \qquad \psi=\frac{f}{2\mu}\Big(\frac{1}{R^2}-\frac{1}{r^2}\Big), \tag{2.6}$$

$$\omega=\frac{\dot f}{2\mu}\Big(\frac{1}{R^2}-\frac{1}{r^2}\Big), \qquad \dot f=\frac{df}{dt}, \qquad f(t)=\frac{a\mu t^2}{R^{-2}-r_0^{-2}}.$$

The resulting solution is valid up to a certain time $t=t_0<t_1$. At a time $t=t_0$, depending on the values of the material parameters, either the material begins to slip or a viscoplastic flow arises in the vicinity of the inner surface. For $\delta\sigma_0<k_0$, the slippage begins before the onset of a viscoplastic flow, so, starting from the time

$$t=t_0=\sqrt{\frac{\delta\sigma_0r_0^2}{a\mu}\Big(\frac{1}{r_0^2}-\frac{1}{R^2}\Big)}$$

the boundary condition (2.3) for $r=r_0$ should be replaced by the condition

$$|\sigma_{r\varphi}|=\delta\,|\sigma_{rr}|+\xi\,|\omega-\omega_{r_0}| \tag{2.7}$$

(ξ is the viscous friction constant). With fulfillment of Eq. (2.7), the heating of the material begins due to the wall friction on the surface $r=r_0$:

$$\theta(r,t_0)=0, \qquad \theta_{,r}(R,t)=0, \qquad \theta(r_0,t)=\alpha_1(\psi(r_0,t)-\psi_{r_0}). \tag{2.8}$$

In accordance with the conditions (2.8), the surface $r = R$ is thermally insulated, $\alpha_1 = \text{const}$ is the heat production constant due to friction, and the heating of the material due to the thermomechanical coupling of reversible deformation and temperature is neglected (the coupling coefficient is assumed to be zero). In this case, Eqs. (2.4), (2.6), and (1.3) imply the relations

$$\sigma_{zz} = -(p + 2\mu) - 2(\varkappa + \mu)g^2 + \xi_1\theta - \xi_2\theta^2 \equiv -\Sigma_1,$$

$$\sigma_{rr} = -\Sigma_1 + 4l\theta g^2, \qquad \sigma_{\varphi\varphi} = -\Sigma_1 + 4\mu g^2, \qquad \sigma_{r\varphi} = 2(\mu - l\theta)g,$$

$$m_{r\varphi} = g, \qquad m_{rr} = -3g^2/2, \qquad m_{\varphi\varphi} = g^2/2,$$

$$\xi_1 = \nu_1 + 6\mu\alpha T_0, \qquad \xi_2 = \nu_3 + 18\mu\alpha^2 + 3\alpha\nu_1 T_0, \qquad l = \nu_1 + \nu_5 + 3\alpha\mu T_0.$$

The heat equation (1.7), the second equation of equilibrium (2.5), and the condition (2.7) yield a system of equations for determining the relative temperature $\theta(r, t)$, the angle of rotation $\psi(r, t)$, and function $f(t)$

$$(1 + \beta_1\theta)\dot{\theta} + \frac{\beta_3 f}{2r^4(\mu - l\theta)^3}\,(lf\dot{\theta} + (\mu - l\theta)\dot{f}) = \lambda(\theta_{,rr} + r^{-1}\theta_{,r}),$$

$$\psi_{,r} = \frac{f}{r^3(\mu - l\theta)}, \qquad \psi_{,t}(r_0, t) = \frac{\delta\sigma_0}{\xi} + \frac{f}{\xi r_0^2} + \omega_{r_0}. \tag{2.9}$$

Equations (2.9) are solved numerically using the conditions (2.8) and the condition $\psi(R, t) = 0$. The calculations continue until the plastic flow condition (1.8) is fulfilled. This condition is satisfied on the surface $r = r_0$ in the form $\sigma_{r\varphi}(r_0, t_*) = -k(\theta(r_0, t_*))$ at a time $t = t_*$ for which the following equation is valid:

$$f(t_*) = -k_0 r_0^2(1 - \theta^2(r_0, t_*)/\theta_m^2).$$

3. VISCOPLASTIC FLOW

Starting at $t = t_*$, an elastoplastic boundary $r = r_1(t)$ moves from the inner boundary surface to the outer surface, and this boundary splits the deformation region into two parts: a reversible (thermoelastic) deformation region $r_1(t) \leqslant r \leqslant R$ (region I) and a viscoplastic flow region $r_0 \leqslant r \leqslant r_1(t)$ (region II).

In the thermoelastic deformation region, Eqs. (2.9) are valid for the relative temperature $\theta^{\mathrm{I}}(r, t)$, the angle of rotation $\psi^{\mathrm{I}}(r, t)$, and function $f(t)$.

Kinematic dependences for the case under consideration have the form

$$\varepsilon_{r\varphi} = \frac{\partial e_{r\varphi}}{\partial t} + \gamma_{r\varphi} = \frac{\partial e_{r\varphi}}{\partial t} + \varepsilon^p_{r\varphi} = \frac{\partial e_{r\varphi}}{\partial t} + \frac{\partial p_{r\varphi}}{\partial t},$$

$$\varepsilon^p_{rr} = \frac{\partial p_{rr}}{\partial t} + 2p_{r\varphi}(x_{r\varphi} + \varepsilon^p_{r\varphi}) - 2p_{r\varphi}\,\frac{\partial\psi}{\partial t},$$

$$\varepsilon^p_{\varphi\varphi} = \frac{\partial p_{\varphi\varphi}}{\partial t} + 2p_{r\varphi}(x_{r\varphi} + \varepsilon^p_{r\varphi}) + 2p_{r\varphi}\,\frac{\partial\psi}{\partial t},$$

$$\varepsilon^p_{rr} = -\varepsilon^p_{\varphi\varphi} = \frac{\varepsilon^p_{r\varphi}(e_{rr} - e_{\varphi\varphi})}{2e_{r\varphi}}.$$

In the viscoplastic flow region, the stresses depending on reversible deformations and temperature are determined from Eqs. (1.4) and (1.6):

$$\sigma_{zz} = -(p + 2\mu) - 2(\mu - (\nu_4 + 3\varkappa\alpha T_0)\theta)m^2_{r\varphi} + 2(\varkappa - (\nu_4 + 3\varkappa\alpha T_0)\theta)(m_{rr} + m_{\varphi\varphi}) + \xi_1\theta - \xi_2\theta^2 = -\Sigma_2,$$

$$\sigma_{rr} = -\Sigma_2 + 2(\mu - l\theta)m_{rr} + (3\mu + l\theta)m^2_{r\varphi}, \tag{3.1}$$

$$\sigma_{\varphi\varphi} = -\Sigma_2 + 2(\mu - l\theta)m_{\varphi\varphi} + (3\mu + l\theta)m^2_{r\varphi}, \qquad \sigma_{r\varphi} = 2(\mu - l\theta)m_{r\varphi}.$$

The following expressions are derived from the associated law of plastic flow (2.9):

$$\sigma_{r\varphi} = -k + \eta\varepsilon^p_{r\varphi}, \qquad \varphi = \varepsilon^p_{r\varphi}(-k + \eta\varepsilon^p_{r\varphi})^{-1}, \qquad \varepsilon^p_{r\varphi} = \frac{1}{\eta}\Big(\frac{f}{r^2} + k_0\Big(1 - \frac{\theta^2}{\theta_m^2}\Big)\Big). \tag{3.2}$$

Stress continuity conditions on the moving elastoplastic boundary $r = r_1(t)$ are used to obtain an equation for determining this boundary:

$$r_1(t) = \sqrt{-f(t)/[k_0(1 - \theta^2(r_1, t)/\theta_m^2)]}\ . \tag{3.3}$$

The heat equation (1.7) for the viscoplastic flow region takes the form

$$(1 + \beta_1\theta^{\rm II})\dot\theta^{\rm II} + \frac{\beta_3 f}{2r^4(\mu - l\theta^{\rm II})^3}\,(lf\dot\theta^{\rm II} + (\mu - l\theta^{\rm II})\dot f) = \lambda(\theta^{\rm II}_{,rr} + r^{-1}\theta^{\rm II}_{,r}) + \frac{1}{2\nu_2\eta}\,\frac{f}{r^2}\Big(\frac{f}{r^2} + k_0\Big(1 - \frac{(\theta^{\rm II})^2}{\theta_m^2}\Big)\Big). \tag{3.4}$$

In accordance with Eqs. (2.1) and (3.2), the following equation is obtained for the angular velocity of the medium points in this region

$$\omega = \psi^{\rm II}_{,rt} = \frac{1}{r^3(\mu - l\theta^{\rm I})}\,(\dot f + fl\theta^{\rm II}_{,t}) + \frac{1}{\eta r}\Big(\frac{f}{r^2} + k_0\Big(1 - \frac{(\theta^{\rm II})^2}{\theta_m^2}\Big)\Big). \tag{3.5}$$

Equations (2.9), (3.3)–(3.5) are supplemented with the boundary conditions

$$\begin{gathered} \theta^{\rm I}_{,r}(R,t) = 0, \qquad \theta^{\rm II}(r_0,t) = \alpha_1(\psi^{\rm II}(r_0,t) - \psi_{r_0}), \\ \psi^{\rm I}(R,t) = 0, \qquad \omega(R,t) = 0, \qquad \theta^{\rm I}(r_1,t) = \theta^{\rm II}(r_1,t), \end{gathered} \tag{3.6}$$

which results in a system of equations for determining the relative temperature $\theta^{\rm I}(r,t)$, $\theta^{\rm II}(r,t)$, the angle of rotation $\psi^{\rm I}(r,t)$, $\psi^{\rm II}(r,t)$, function $f(t)$, and the elastoplastic boundary $r = r_1(t)$.

For the numerical implementation of problem (2.9), (3.4)–(3.6), two time-varying grids with respect to variable r are constructed:

— in the thermoelastic deformation region: $r = r_{1\,i+1} + h^e_{i+1}j$, $j = \overline{0, N^e - 1}$, $h^e_{i+1} = (1 - r_{1\,i+1})/N^e$;

— in the viscoplastic flow region: $r = r_0 + h^p_{i+1}j$, $j = \overline{1, N^p}$, $h^p_{i+1} = (r_{1\,i+1} - r_0)/N^p$. At each time step $t = t_* + dt(i+1)$, $i = \overline{0, N}$, due to the movement of the elastoplastic boundary, the grid changes. The temperature and angle of rotation for the newly constructed grid at the previous time step are calculated using interpolation.

The resulting values of the relative temperature $\theta^{\rm I}(r,t)$ and $\theta^{\rm II}(r,t)$, the angle of rotation $\psi^{\rm I}(r,t)$ and $\psi^{\rm II}(r,t)$, and function $f(t)$ are used to determine the distributions of the angular velocity ω, stress $\sigma_{r\varphi}$, thermoelastic strain components $m_{r\varphi}$, and plastic strain components $p_{r\varphi}$. The unknown components of the reversible deformations m_{rr} and $m_{\varphi\varphi}$ and of the irreversible deformations p_{rr} and $p_{\varphi\varphi}$ are calculated from the system of equations

$$\frac{\partial p_{\varphi\varphi}}{\partial t} = -\varepsilon^p_{r\varphi}\,\frac{p_{\varphi\varphi} - m^2_{r\varphi}}{m_{r\varphi}} + \frac{4\varepsilon_{r\varphi}p_{r\varphi}}{2 + m^2_{r\varphi}}\Big(1 + m_{\varphi\varphi} - \frac{1}{2}\,m^2_{r\varphi} - 2m_{r\varphi}p_{r\varphi}\Big),$$

$$m_{rr} = p_{\varphi\varphi} - 3m^2_{r\varphi}/2 - 2m_{r\varphi}p_{r\varphi}, \qquad p_{rr} + p_{\varphi\varphi} = -2p^2_{r\varphi}, \qquad m_{rr} + m_{\varphi\varphi} = -m^2_{r\varphi}.$$

Next, the diagonal stress components and the additional hydrostatic pressure are determined from the first equilibrium equation (2.5) and expressions (3.1).

4. DEFORMATION AT A CONSTANT AND DECREASING ROTATION VELOCITY

System (2.9), (3.4)–(3.6) is valid at a constant rotation velocity, starting from $t = t_1$. In this system, $\psi(r_0,t) = at_1t - at_1^2/2$ according to Eqs. (2.2). In this case, the viscoplastic flow region continues to increase.

It is assumed that, since $t = t_2 > t_1$, the angular velocity of the inner surface decreases: $\omega_{r_0}(t) = at_1 - b(t - t_2)$. In this case, the viscoplastic flow region first increases and then, since $t' > t_2$, begins decreasing. From a time t'', a new elastoplastic boundary $r = r_2(t)$ is formed, which moves from the surface $r = r_1(t')$ to the inner surface $r = r_0$ and separates the decreasing flow region $r_0 \leqslant r \leqslant r_2(t)$ from the region $r_2(t) \leqslant r \leqslant r_1(t')$, in which irreversible deformations do not accumulate (the irreversible strain tensor does not change with time). In the region $r_1(t') \leqslant r \leqslant R$, the deformation is reversible.

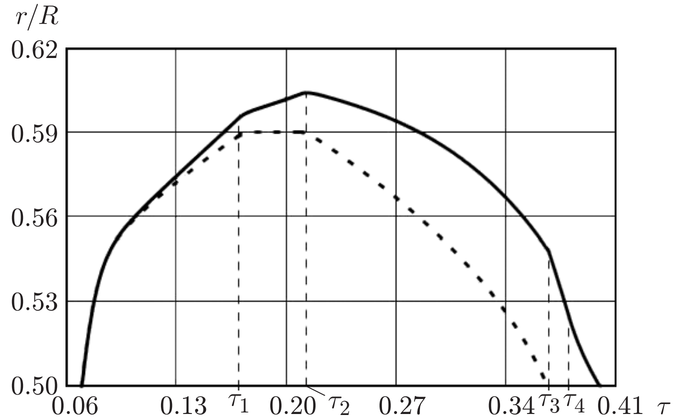

Fig. 1. Change in the viscoplastic flow region during deformation: with account for heating (solid line) and isothermal case (dashed line).

In the reversible deformation regions $r_1(t') \leqslant r \leqslant R$ and $r_2(t) \leqslant r \leqslant r_1(t')$, the heat equation in system (2.9) is valid with the additional condition that the values of function θ match on the unchanged boundary $r = r_1(t')$. In the viscoplastic flow region, Eq. (3.4) is still fulfilled. The temperature, the rotation angle, and functions $f(t)$ and $r = r_2(t)$ are determined by constructing a system of equations similar to system (2.9), (3.3)–(3.6).

At a time $t = t_3$, with a continuing decrease in the rotation velocity of the inner rigid surface, slippage stops and then the adhesion condition is fulfilled. An instant $t = t_3$ is found from the condition $f(t_3) = -\delta\sigma_0 r_0^2$. Starting from a time $t = t_3$, the material cools down, and the boundary conditions on the surface $r = r_0$ take the form

$$\psi(r_0) = \psi(r_0, t_3), \qquad \theta(r_0, t) = \theta(r_0, t_3)(1 - \alpha_2(t - t_3)).$$

For the selected problem parameters, first, for $t = t_4 = ab^{-1}t_1 + t_2$, the speed of the inner surface becomes zero, then, for $t = t_5$, the boundary surface $r = r_2(t)$ reaches the rigid wall $r = r_0$, and two regions of reversible deformation remain in the material: $r_1(t') \leqslant r \leqslant R$ (thermoelastic) and $r_0 \leqslant r \leqslant r_1(t')$ (area with accumulated irreversible deformations). This moment of time is found by solving the equation

$$r_0^2 k_0 (1 - \theta^2(r_0, t_5)/\theta_m^2) = -f(t_5).$$

From the moment of time $t = t_5$ the process of temperature equalization in the cylindrical layer due to thermal conductivity is described by the first equation (2.9) and continues after the moment of time

With the selected parameters of the problem, the velocity of the inner surface becomes equal to zero at $t = t_4 = ab^{-1}t_1 + t_2$ and then the boundary surface $r = r_2(t)$ reaches the rigid wall $r = r_0$ at $t = t_5$, and two reversible deformation regions remain in the material: $r_1(t') \leqslant r \leqslant R$ (thermoelastic) and $r_0 \leqslant r \leqslant r_1(t')$ (region with accumulated irreversible deformations). This instant is found from the solution of the equation

$$r_0^2 k_0 (1 - \theta^2(r_0, t_5)/\theta_m^2) = -f(t_5).$$

Since $t = t_5$, the temperature equalization process in the cylindrical layer due to thermal conductivity is described by the first equation in system (2.9) and continues after the time $t = t_6 = t_4 + \alpha_2^{-1}$,, when $\theta(r_0, t_6) = 0$.

Figure 1 shows the change in the position of the elastoplastic boundary during the entire deformation process, and in Figs. 2 and 3 illustrate the distributions of the relative temperature θ and the angle of rotation ψ at different times. The calculations are carried out in dimensionless variables

$$\tilde{r} = \frac{r}{R}, \qquad \tau = \alpha t, \qquad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\mu}$$

with the following parameter values: $a\eta/\mu^2 = 0.004$, $r_0/R = 0.5$, $k/\mu = 0.006\,21$, $a/b = 1$, $a\xi/\mu = 0.005$, $\beta_1 = 0.5$, $\beta_3 = -0.5$, $\delta\sigma_0/\mu = 0.005$, $l/\mu = 0.001$, $\nu_1/\mu = 0.02$, $\alpha_1 = 100$, $\alpha_2 = 50$, $q/R^2 = 10$.

CONCLUSIONS

This work describes the solution to the problem of the viscometric flow of an elastoviscoplastic material in the gap between two rigid coaxial cylindrical surfaces, with possible slippage of the material and its heating

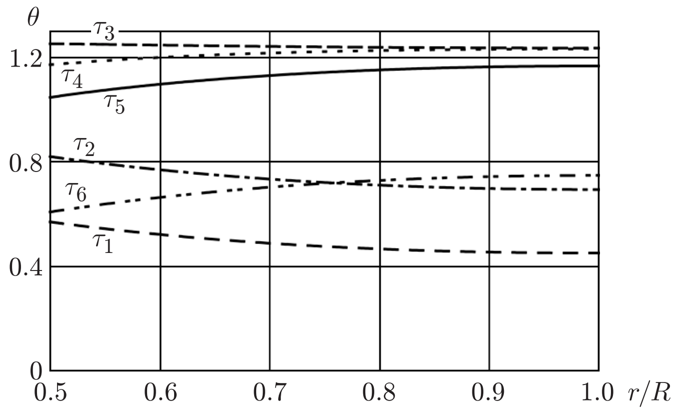

Fig. 2. Temperature distribution at different times.

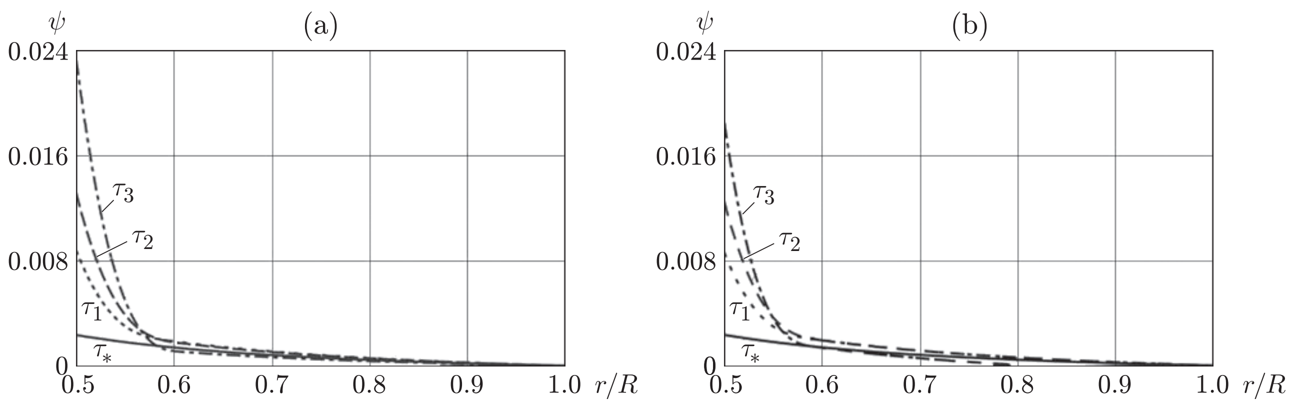

Fig. 3. Distribution of the values of the angle of rotation at different times in the case of deformation with account for heating (a) and in the isothermal case (b).

due to friction against the cylinder walls on one of the surfaces. The change in the viscoplastic flow region differs significantly from the change in the region in the isothermal case. In the case under consideration, as the rotation velocity of the inner cylinder increases, the viscoplastic flow region develops faster, the viscoplastic flow region continues to increase at a constant velocity. In the isothermal case, it increases mildly in the isothermal case and does not develop further. With a decreasing rotation velocity, the viscoplastic flow region becomes smaller at a much slower rate than in the isothermal case. Significant differences in the displacement distribution are observed with a decreasing rotational velocity: they are greater than in the isothermal case.

REFERENCES

1. V. P. Myasnikov, "Some Exact Solutions for Rectilinear Movements of a Viscoplastic Medium," Prikl. Mekh. Tekh. Fiz. **2** (2), 54–60 (1961).
2. G. I. Bykovtsev and A. D. Chernyshov, "Viscoplastic Flow in Non-Circular Cylinders in the Presence of a Pressure Drop," Prikl. Mekh. Tekh. Fiz. **5** (4), 94–96 (1964).
3. P. M. Ogibalov and A. Kh. Mirzadzhanzade, *Unsteady Motion of Viscoplastic Media* (Izd. Mosk. Gos. Univ., Moscow, 1970) [in Russian].
4. D. V. Georgievskii, "Rigid Zones in Statically Determinate and Indeterminate Problems of a Viscoplastic Flow," in *Problems of Mechanics of Deformable Solids and Rocks: Collection of Papers for the 75th Anniversary of E. I. Shemyakin* (Fizmatlit, Moscow, 2006) [in Russian].

5. A. V. Rezunov and A. D. Chernyshov, “The Problem of Pure Shear of a Viscoplastic Material Between Two Cylindrical Surfaces,” in *Mechanics of a Deformable Solid* (Kuybishev State University, Kuybyshev, 1975) [in Russian].
6. G. Duvaut and J.-L. Lions, *Les inequations en mecanique et en physique* (Dunod, Paris, 1972).
7. P. P. Mosolov and V. P. Myasnikov, *Mechanics of Rigid Plastic Media* (Nauka, Moscow, 1981) [in Russian].
8. D. D. Ivlev, “Determining Displacements in Elastoplastic Problems,” in *Advances in Mechanics of Deformable Media. Dedicated to the 100th Anniversary of B. G. Galerkin* (Nauka, Moscow, 1975) [in Russian].
9. G. P. Cherepanov and B. D. Annin, *Elastic-Plastic Problems* (The American Society of Mechanical Engineers, 1988).
10. L. A. Galin, *Elastic-Plastic Problem* (Nauka, Moscow, 1984) [in Russian].
11. D. D. Ivlev, *From the History of Discussions in Mechanics. Three Discussions* (Belarusian National Technical University, Minsk, 2012) [in Russian].
12. E. H. Lee, “Elastic-Plastic Deformation at Finite Strains,” Trans ASME. Ser. E. J. Appl. Mech. **36** (1), 1–6 (1969).
13. V. I. Kondaurov, “Equations of Elastoviscoplastic Medium with Finite Deformations,” Prikl. Mekh. Tekh. Fiz. **23** (4), 133–139 (1982) [Journal of Applied Mechanics and Technical Physics **23** (2), 584–591 (1982)].
14. V. I. Levitas, *Large Elastoplastic Deformations of Materials at High Pressure* (Naukova Dumka, Kiev, 1987) [in Russian].
15. B. D. Annin and S. N. Korobeinikov, “Admissible Forms of Elastic Laws of Deformation in the Constitutive Relations of Elastoplasticity,” Sib. Zhurn. Industr. Matematiki **1** (1), 21–34 (1998).
16. A. D. Chernyshov, “Constitutive Relations for an Elastoplastic Body at Finite Deformations,” Izv. Ros. Akad. Nauk. Mekhanika Tverdogo Tela, No. 1, 120–128 (2000).
17. A. A. Rogovoi, “Constitutive Relations for Finite Elastic-Inelastic Strains,” Prikl. Mekh. Tekh. Fiz. **46** (5), 138–149 (2005) [Journal of Applied Mechanics and Technical Physics **46** (5), 730–739 (2005].
18. G. I. Bykovtsev and A. V. Shitikov, “Finite Deformations of Elastoplastic Media,” Dokl. Akad. Nauk SSSR **311** (1), 59–62 (1990).
19. A. A. Burenin, G. I. Bykovtsev, and L. V. Kovtanyuk, “One Simple Model for an Elastoplastic Medium Under Finite Deformations,” Dokl. Ros. Akad. Nauk. **347** (2), 199–201 (1996).
20. V. P. Myasnikov, “Equations of Motion of Elastoplastic Materials at Large Deformations,” Vestnik Dal’nevostochnogo Otdeleniya Ros. Akad. Nauk, No. 4, 8–13 (1996).
21. A. A. Burenin and L. V. Kovtanyuk, “Large Irreversible Deformations and Elastic Aftereffect,” (Dal’nauka, Vladivostok, 2013) [in Russian].
22. A. S. Begun, A. A. Burenin, and L. V. Kovtanyuk, “Helical Viscoplastic Flow in the Gap Between Rigid Cylinders,” Izv. Ros. Akad. Nauk. Mekhanika Tverdogo Tela, No. 6, 55–70 (2017).
23. A. A. Burenin, L. V. Kovtanyuk, and A. S. Ustinova, “Accounting for the Elastic Properties of a Non-Newtonian Material Under Its Viscosimetric Flow,” Prikl. Mekh. Tekh. Fiz. **49** (2), 143–151 (2008) [Journal of Applied Mechanics and Technical Physics **49** (2), 277–284 (2008)].
24. A. A. Burenin, L. V. Kovtanyuk, and A. S. Ustinova, “Viscometric Flow of an Incompressible Elastic-Viscoplastic Material in the Presence of Lubricant on Boundary Surfaces,” Sib. Zhurn. Industr. Matematiki **15** (2), 43–55 (2012).
25. A. S. Begun, A. A. Burenin, and L. V. Kovtanyuk, “Flow of an Elastoviscoplastic Material Between Rotating Cylindrical Surfaces with Nonrigid Cohesion,” Prikl. Mekh. Tekh. Fiz. **56** (2), 146–158 (2015) [Journal of Applied Mechanics and Technical Physics **56** (2), 293–303 (2015)].
26. L. V. Kovtanyuk, “Modeling of Large Elastoplastic Deformations in an Nonisothermal Case,” Dal’nevostochnyi Matematicheskii Zhurnal **5** (1), 110–120 (2004).
27. L. V. Kovtanyuk and A. V. Shitikov, “Theory of Large Elastoplastic Deformations with Account for Temperature and Rheological Effects,” Vestnik Dal’nevostochnogo Otdeleniya Rossiiskoi Akademii Nauk, No. 4, 87–93 (2006).

28. A. A. Burenin, L. V. Kovtanyuk, and G. L. Panchenko, “Nonisothermal Motion of an Elastoviscoplastic Medium Through a Pipe Under a Changing Pressure Drop,” Doklad Akademii Nauk **464** (3), 284–287 (2015) [Doklady Physics **60** (9), 419–422 (2015)].
29. A. A. Burenin, L. V. Kovtanyuk, and G. L. Panchenko, “Development and Deceleration of Viscoplastic Flow in a Layer Heated by Friction on a Rough Plane,” Prikl. Mekh. Tekh. Fiz. **56** (4), 101–111 (2015) [Journal of Applied Mechanics and Technical Physics **56** (4), 626–635 (2015].
30. A. A. Burenin, L. V. Kovtanyuk, and G. L. Panchenko, “The Motion of an Elastoplastic Medium in a Circular Tube Heated Due to Boundary Friction,” Prikladnaya Matematika I Mekhanika **80** (2), 265–275 (2016) [Journal of Applied Mathematics and Mechanics **80** (2), 190–197 (2016)].
31. A. A. Burenin, L. V. Kovtanyuk, and G. L. Panchenko, “Deformation and Heating of an Elastoviscoplastic Cylindrical Layer Moving Owing to a Varying Pressure Drop,” Izvestiya Akademii Nauk, Mekhanika Tverdogo Tela, No. 1, 6–18 (2018) [Mechanics of Solids **53** (1), 1–11 (2018)].
32. V. A. Znamenskii and D. D. Ivlev, “Equations of a Viscoplastic Body with Piecewise Linear Potentials,” Izv. Akad. Nauk SSSR. Otd. Tekh. Nauk. Mekhanika I Mashinostroenie, No. 6, 114–118 (1963).