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Residual stresses calculation in a thermoelastoplastic torus after cooling

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Abstract. The present study deals with the boundary value problems under toroidal symmetry conditions. The residual stresses after cooling (unloading) in an elasto-plastic material are calculated. Throughout the paper the conventional Prandtl-Reuss model is generalised and used. The solution to the problem of hollow torus cooling under a temperature gradient is obtained and discussed. Analytical solutions, as an approximation of complete boundary value problem, describing residual deformations and stresses under conditions of toroidal symmetry are constructed and discussed.

Preliminary remarks

Requirement for the use of lightweight parts and structures has significantly increased in many branches of modern mechanical engineering and aircraft construction. This problem is partially solved by the use of functionally gradient materials (for example, titanium alloys) [1–5]. Functionally graded material is a class of modern materials with different properties depending on the characteristic microstructural size. In nature, functionally graded materials are bones, teeth, etc. One of the unique characteristics of functionally graded materials is their ability to adapt to specific operational loads.

Another topical problem is the quick and easy replacement of failed parts. The processes of additive manufacturing have undoubted advantages in the case of their replacement. These production methods include physical or chemical liquid gas deposition, plasma spraying, self-propagating high temperature synthesis, powder metallurgy, centrifugal casting, and laser metal deposition. The laser metal deposition process is a class of additive manufacturing processes that allows a functional part to be produced directly from a 3D computer model and possibly from the various materials.

Products produced by such methods are more economically profitable, and the production is less toxic in comparison with other technological processes. Nevertheless, products and materials obtained by the additive method often exhibit the microstructural features and are functionally graded materials.

Mathematical models of deformation of the items manufactured by the methods described above must undoubtedly take into account temperature effects. The thermoelasticity model obtained by generalising the classical Prandtl-Reuss model fully meets the requirements of modern engineering for researchers. Earlier, the authors of this research have solved a number of boundary value problems for temperature stresses calculation in the bodies with axial and central symmetry [6–18]. In this work, we will consider the problem of the residual stresses calculation under conditions of toroidal symmetry. The basis for the plastic flow calculation preceding the stage of material unloading is the results presented in publications [19–25].

1. Differential equations of thermoelastoplastic model

Transformation from Cartesian $(X,\ Y,\ Z)$ to the toroidal (r,θ,φ) coordinates is given by relations:

$$X = \Omega\cos\varphi, \quad Y = \Omega\sin\varphi, \quad Z = R_0\cos\theta, \quad \Omega = R_0 + r\sin\theta, \tag{1}$$

where R_0 is the major radius of the torus, $r \in [r_1, r_2]$, r_1 and r_2 are the inner and outer radius of the toroidal surface. The center of the torus corresponds to the origin of the Cartesian coordinates and the center of the toroidal system is located on the generatrix of the torus.

Strain tensor components are the sum of the thermoelastic e_{ij} and the plastic p_{ij} parts.

$$d_{ij} = e_{ij} + p_{ij} \tag{2}$$

The strain tensor components depend on the displacement vector u_i by following equations

$$\begin{gathered} d_{\theta\theta} = \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r}, \quad d_{\varphi\varphi} = \frac{u_r\sin\theta + u_\theta\cos\theta}{\Omega} + \frac{u_{\varphi,\varphi}}{\Omega}, \quad d_{r\theta} = \frac{1}{2}\left(\frac{u_{r,\theta}}{r} + u_{\theta,r} - \frac{u_\theta}{r}\right), \\ d_{rr} = u_{r,r}, \quad d_{r\varphi} = \frac{1}{2}\left(\frac{u_{r,\varphi}}{\Omega} + u_{\varphi,r} - \frac{u_\varphi\sin\theta}{\Omega}\right), \quad d_{\theta\varphi} = \frac{1}{2}\left(\frac{u_{\theta,\varphi}}{\Omega} + u_{\varphi,\theta} - \frac{u_\varphi\cos\theta}{\Omega}\right). \end{gathered} \tag{3}$$

There comma denotes the partial derivatives with respect to the corresponding spatial coordinate.

In the toroidal coordinate net the equilibrium equations take the form

$$\begin{gathered} \sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{r\varphi,\varphi}}{\Omega} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\sin\theta}{\Omega}(\sigma_{rr} - \sigma_{\varphi\varphi} + \cot\theta\sigma_{r\theta}) = 0, \\ \sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{\sigma_{\theta\varphi,\varphi}}{\Omega} + \frac{2\sigma_{r\theta}}{r} + \frac{\sin\theta}{\Omega}(\sigma_{r\theta} + \cot\theta(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})) = 0, \\ \sigma_{r\varphi,r} + \frac{\sigma_{\theta\varphi,\theta}}{r} + \frac{\sigma_{\varphi\varphi,\varphi}}{\Omega} + \frac{\sigma_{r\varphi}}{r} + \frac{2\sin\theta}{\Omega}(\sigma_{r\varphi} + \cot\theta\,\sigma_{\theta\varphi}) = 0. \end{gathered} \tag{4}$$

The constitutive equations of the thermoelastic continuum can be assumed in the form of the Duhamel-Neumann's law:

$$\sigma_{ij} = \lambda\delta_{ij}\mathrm{tr}e_{ij} - \alpha\delta_{ij}(3\lambda + 2\mu)(T - T_0) + 2\mu e_{ij}, \tag{5}$$

where δ_{ij} is Kronecker delta, λ, μ are Lame constants, α is coefficient of the linear thermal expansion, $(T - T_0)$ is the difference between the initial T_0 and the current temperature T.

Note that in the further consideration, we will neglect the influence of deformation processes on the change in the temperature field. In the toroidal coordinates the heat equation reads by

$$T_{,rr} + \frac{(R_0 + 2r\sin\theta)T_{,r}}{r(R_0 + r\sin\theta)} + \frac{T_{,\theta\theta}}{r^2} + \frac{\cos\theta\, T_{,\theta}}{r(R_0 + r\sin\theta)} + \frac{T_{,\varphi\varphi}}{(R_0 + r\sin\theta)^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}. \tag{6}$$

Under the given boundary conditions and known distributions of irreversible deformations p_{ij}, the system of equations (3)–(6) specifies the evolution of the stress-strain state with thermal impact of the toroidal solid in the toroidal coordinates.

2. Statement of boundary value problem

Consider hollow torus with the major radius R_0 and $r_1 < r < r_2$. Thermal influence is given by axisymmetric (according to Z-axis) temperature distribution. In this case, the stress-strain state does not depend on angular coordinate φ. Thus, the following components of the displacement vector, the strain tensor and the stress tensor will be equal to zero:

$$u_\varphi = 0, \quad d_{r\varphi} = d_{\theta\varphi} = 0, \quad \sigma_{r\varphi} = \sigma_{\theta\varphi} = 0. \tag{7}$$

On the outer surface $r = r_2$, we define the state of free thermal expansion according to the boundary conditions

$$\sigma_{rr}(r_1, \theta) = 0, \quad \sigma_{r\theta}(r_1, \theta) = 0, \quad \sigma_{rr}(r_2, \theta) = 0, \quad \sigma_{r\theta}(r_2, \theta) = 0. \tag{8}$$

Consider the solution of the stationary heat equation (6) with the boundary conditions:

$$T(r_1, \theta) = T_k, \quad T(r_2, \theta) = T_0. \tag{9}$$

Numerical analysis of solutions to the heat equation showed that the calculated temperature distribution depends significantly on the geometry of the torus and for small values of the parameter $\epsilon = r_2/R_0$ it can be described by a function that depends only on the radial coordinate. As $\epsilon = r_2/R_0$ tends to zero, the toroidal symmetry turns into cylindrical, which allows one-dimensional analytical solutions to be taken with a sufficient degree of accuracy in the approximation of the hypothesis of generalized plane deformation. With this approach, it is important to determine the admissible finite values of the parameter ϵ, for which the cylindrical solutions will satisfactorily describe two-dimensional numerical solutions in toroidal coordinates.

The stationary heat conduction equation at $\epsilon = 0$ (under conditions of axial symmetry) takes the form:

$$T_{,r} + rT_{,rr} = 0. \tag{10}$$

Numerical experiments have shown that the maximum deviation of the analytical solution of the equation (10) from the numerical solution of the equation (6) is less than 2% for $\epsilon = 0.1$ and $r_1/r_2 = 0.4$. Therefore, with a sufficiently high degree of accuracy, the temperature distribution at $\epsilon < 0.1$ can be considered for the one-dimensional case.

For $\epsilon = 0$ we have the equilibrium equations and the relations for the strains:

$$\begin{aligned} &\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{2\sigma_{r\theta}}{r} = 0. \\ &d_{rr} = F_{,r} \quad d_{\varphi\varphi} = C, \quad d_{\theta\theta} = \frac{F}{r}, \quad d_{r\theta} = 0, \end{aligned} \tag{11}$$

where $F(r)$ is the unknown function, C is the unknown constant. In this case, the components of the displacement vector can be represented as:

$$u_r(r, \theta) = F(r) + R_0 C \sin\theta, \quad u_\theta(r, \theta) = R_0 C \cos\theta. \tag{12}$$

The form of the function $F(r)$ depends on the stress-strain state and it is determined by taking into account the presence or absence of plastic flow in a given area of the material.

3. Stress-strain state under accumulated irreversible strains

As early shown in papers [19–21], with free thermal expansion, the temperature gradient specified by the conditions (9) leads to the appearance of several regions of deformation in the material: two regions of plastic flow corresponding to the edge and face of the Tresca prism, and a region

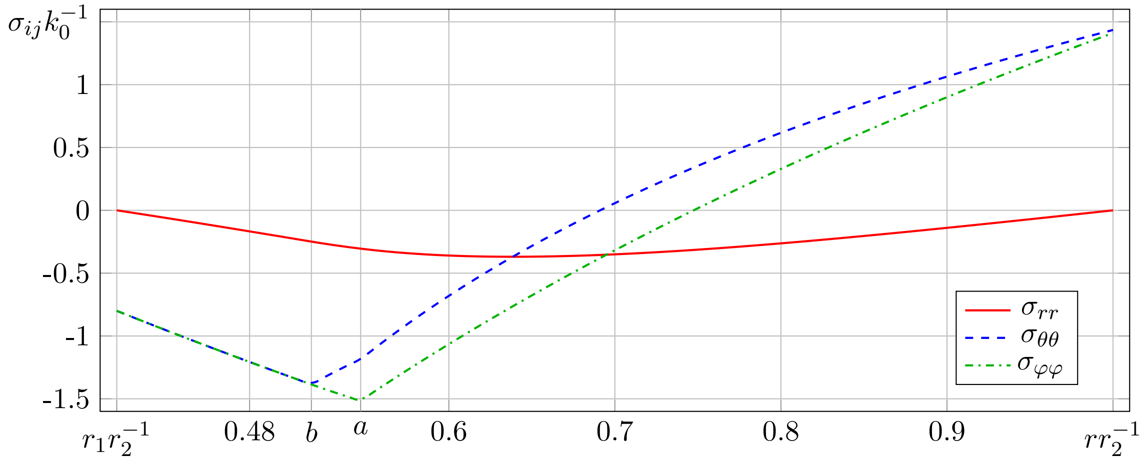

Figure 1. Thermal stresses in the toroidal coordinates, $r_1r_2^{-1} = 0.4$, $\epsilon = 0$, $br_2^{-1} = 0.517$, $ar_2^{-1} = 0.547$.

of thermoelastic deformation. Solutions in each area of a specific area are given in [19–21] and differ from each other by the form of the function $F(r)$. Distributions of the thermal stresses in the case of plastic flow are shown on the figure 1.

Consider the cooling process of the torus from the stress-strain state showing on figure 1 to the state when the temperature field returns to the initial distribution ($T = T_0$). In this case, after the plastic flow, the unloading process will begin, characterised by thermoelastic deformation taking into account the accumulated irreversible deformations. Expressing one of the components of plastic deformations in terms of two others ($p_{\varphi\varphi} = -p_{rr} - p_{\theta\theta}$), we obtain the final equations for stresses forming in the material during unloading:

$$\begin{aligned}
&\sigma_{rr} = \frac{2\mu}{\eta^2}\int_{r_1}^{r}\frac{p_{rr}(\rho)-p_{\theta\theta}(\rho)}{\rho}d\rho - \frac{2\mu^2}{(\lambda+2\mu)r^2}\int_{r_1}^{r}\rho[p_{rr}(\rho)+p_{\theta\theta}(\rho)]d\rho + \frac{Q}{r^2} + P,\\
&\sigma_{\theta\theta} = [r\sigma_{rr}(r)]_{,r}, \quad \sigma_{\varphi\varphi} = \mu\gamma[p_{rr}(r)+p_{\theta\theta}(r)] + \frac{\lambda\sigma_{rr}(r)+\lambda\sigma_{\theta\theta}(r)}{2(\lambda+\mu)}.
\end{aligned} \tag{13}$$

Here, P, Q are integration constants.

According to the plasticity conditions, plastic deformations [19–21] can be represented as:

$$p_{rr} = \begin{cases} p_{rr}^*, & r_1 \le r \le b,\\ p_{rr}^{**}, & b \le r \le a,\\ 0, & r_1 \le a \le r_2, \end{cases} \qquad p_{\theta\theta} = \begin{cases} p_{\theta\theta}^*, & r_1 \le r \le b,\\ 0, & b \le r \le a,\\ 0, & r_1 \le a \le r_2. \end{cases} \tag{14}$$

The plastic deformations are described by the formulas in the domain $r_1 \le r \le b$:

$$\begin{aligned}
&p_{rr}^* = -\frac{C}{2} + \frac{D}{r^2} - \omega\int_{r_1}^{r}\frac{k(\rho)}{\rho}d\rho + \frac{\omega}{r^2}\int_{r_1}^{r}k(\rho)\rho d\rho - \frac{3}{r^2}\int_{r_1}^{r}\Delta(\rho)\rho d\rho - 2\frac{k}{\mu\gamma} + 2\Delta,\\
&p_{\theta\theta}^* = -\frac{C}{2} - \frac{D}{r^2} - \omega\int_{r_1}^{r}\frac{k(\rho)}{\rho}d\rho - \frac{\omega}{r^2}\int_{r_1}^{r}k(\rho)\rho d\rho + \frac{3}{r^2}\int_{r_1}^{r}\Delta(\rho)\rho d\rho + \frac{k}{\mu\gamma} - \Delta,\\
&p_{\varphi\varphi}^* = C + 2\omega\int_{r_1}^{r}\frac{k(\rho)}{\rho}d\rho + \frac{k}{\mu\gamma} - \Delta, \quad \omega = \frac{1}{3\lambda+2\mu},
\end{aligned} \tag{15}$$

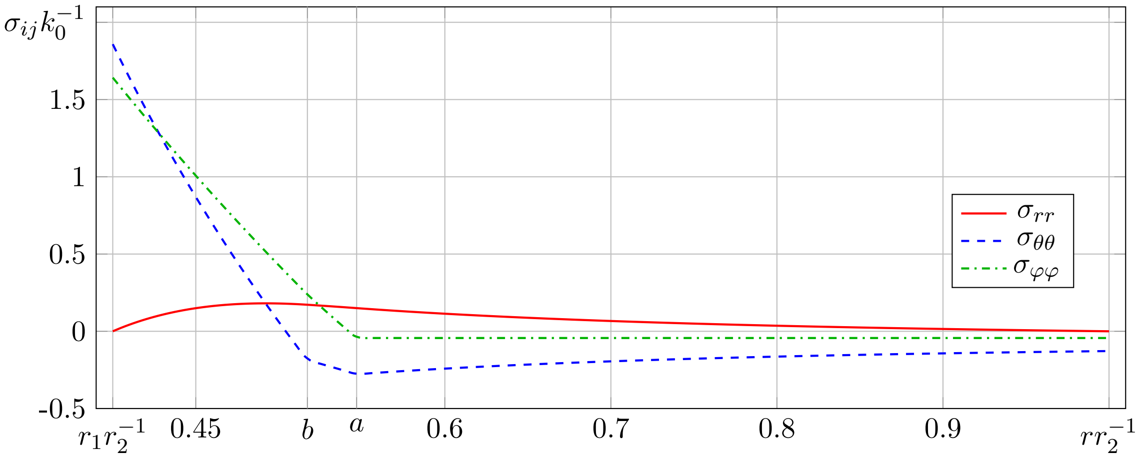

Figure 2. Residual stresses, $r_1r_2^{-1} = 0.4$, $\epsilon = 0$, $br_2^{-1} = 0.517$, $ar_2^{-1} = 0.547$.

and in the domain $b \leq r \leq a$:

$$F^{**}(r) = \frac{\psi}{2\eta}\left[\frac{\eta+1}{r^\eta}\int_{r_1}^{r}\Delta(\rho)\rho^\eta d\rho + (\eta-1)r^\eta\int_{r_1}^{r}\frac{\Delta(\rho)}{\rho^\eta}d\rho\right] + rC$$
$$- \frac{1}{2(\lambda+\mu)}\left[\frac{1}{r^\eta}\int_{r_1}^{r}k(\rho)\rho^\eta d\rho + r^\eta\int_{r_1}^{r}\frac{k(\rho)}{\rho^\eta}d\rho\right] + Mr^\eta + \frac{N}{r^\eta}, \tag{16}$$
$$p_{rr}^{**} = \frac{1}{2}\left(F_{,r}^{**} - C - \frac{k}{\mu}\right), \quad p_{\varphi\varphi}^{**} = \frac{1}{2}\left(C + \frac{k}{\mu} - F_{,r}^{**}\right), \quad \psi = \frac{3\lambda+2\mu}{\lambda+\mu},$$

where C, D are the constants, Δ is the difference between the maximum temperature in the each point of the material and the initial temperature.

Equations (12)–(16) determine the stress-strain state under conditions of elastic unloading of the material. Distributions of the residual stresses after torus cooling are shown on the figure 2. Note that, in this case, the zones of plastic deformation are not sufficient for the occurrence of repeated plastic flow.

4. Conclusion

The paper is devoted to the boundary value problems under toroidal symmetry conditions. The residual stresses after cooling (unloading) in an elasto-plastic material have been calculated. Throughout the paper the conventional Prandtl-Reuss model has been generalised and used. The solution to the problem of hollow torus cooling under a temperature gradient has been obtained and discussed. Analytical solutions, as an approximation of complete boundary value problem, describing residual deformations and stresses under conditions of toroidal symmetry have constructed and discussed. The present analytical solutions can be applied to the calculation thick-walled long constructions manifesting symmetry close to tororidal one.

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