Applied Mechanics and Materials Vols. 635-637 (2014) pp 3-6
Submitted: 10.07.2014
Accepted: 22.07.2014
doi:10.4028/www.scientific.net/AMM.635-637.3

Analysis of 2-D Impedance Cloaking Problem Based on Boundary Element Method

Gennady V. Alekseev[1,2,3, a], Andrei Baydin[1,b] and Olga Larkina[1,c]

[1]Far Eastern Federal University, 8, Suhanova St., Vladivostok, 690950, Russia

[2]Institute of Applied Mathematics FEB RAS, 7, Radio St., Vladivostok, 690041, Russia

[3]Vladivostok State University of Economics and Service, 41, Gogolya St., Vladivostok, 690014, Russia

[a]alekseev@iam.dvo.ru, [b]thulfm@gmail.com, [c] larkina-olga@rambler.ru

Keywords: Helmholtz equation, boundary control, impedance, control problem, boundary element method.

Abstract. Control problems are considered for a two-dimensional model describing wave scattering in an unbounded homogenous medium containing an impenetrable covered (cloaked) boundary. The control is a surface impedance which enters the boundary condition as a coefficient. The solvability of the original scattering problem for 2-D Helmholtz equation and of the control problem is proved. Optimality system drescribing the necessary extremum conditions are derived. The algorithm for numerical solving of the control problem based on the optimality system and boundary element method is designed.

Introduction

It was proposed that perfect invisibility cloaking shells can be constructed for hiding objects from electromagnetic illumination [1]. However, the difficulty in fabricating such shells stems from the requirements on the material that compose it. Cloak obtained through such technique has anisotropic, spatially varying optical constants. In addition, some of the material parameters have infinite values at the interior surface of the cloak. In order to facilitate easier realization as well as to avoid infinities in optical constants, cloaks with simplified material parameters were proposed. Hence, perfect hiding with such a simplified cloak is not possible [2]. Another approach in cloaking material bodies consists of coating it's outer boundary with special material having the certain value of surface impedance. In this case, the cloaking problem is reduced to choosing the impedance such that the wave scattered by the object have certain properties [3,4]. We mention papers [5-10] devoted to development of methods of solving impedance cloaking problems based on optimization approach of solving inverse problems for the wave equations. It should be noted also papers [11-13] devoted to applying optimization methods for solving related problems of technical gas dynamics.

Formulation and Solvability of the Original Scattering Problem

Let Ω be bounded region in R^2 with a connected complement $\Omega^c = R^2 \setminus \Omega$ and with a boundary Γ. It is well known that the problem of scattering waves in a homogenous medium containing an impermeable obstacle Ω with a covered boundary is reduced to finding a function $u = u^{inc} + u^s$ in Ω^c that satisfy the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^c, \tag{1}$$

and obey the impedance boundary condition and the Sommerfield radiation condition in R^2

$$\partial u / \partial n + i\, k\, \lambda\, u = 0 \quad \text{on } \Gamma, \quad \lim_{r \to \infty} \sqrt{r}\,(\partial u^s / \partial r - i\, k\, u^s) = 0 \quad \text{as } r = |x| \to \infty. \tag{2}$$

Here u^{inc} is the incident wave, u^s is the scattered wave, λ is the surface impedance on boundary Γ, k is the wave number, i is the imaginary unit, n is the outward (relative to Ω) unit normal vector.

The following conditions are assumed below to hold:
(i) Ω is a bounded domain in R^2 with a connected complement Ω^c and with a boundary Γ in $C^{0,1}$.

Let us introduce the function spaces to be used in the analysis of problem Eq. 1 – Eq. 3. Let B_R be a disk of radius R containing Ω, and let $\Omega_e = \Omega^c \cap B_R$. Clearly, Ω_e is a bounded domain in R^2 with the boundary $\partial\Omega_e$ consisting of two parts: Γ and Γ_R where Γ_R is a boundary of B_R. For any open subset Q in B_R, we use the Sobolev space $H^1(Q)$ of complex or real scalar functions defined in Q. We also use the trace spaces $H^{1/2}(\partial Q)$ and in particular $H^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma_R)$. Let $H^{-1/2}(\partial Q)$ denote the dual of the space $H^{1/2}(\partial Q)$ with respect to $L^2(\partial Q)$. The norms in $H^1(Q)$, $H^{1/2}(\partial Q)$, and $H^{-1/2}(\partial Q)$ are denoted by $\|\cdot\|_{1,Q}$, $\|\cdot\|_{1/2,\partial Q}$, $\|\cdot\|_{-1/2,\partial Q}$ respectively. The inner products and norms in $L^2(Q)$ and $L^2(\partial Q)$ are designated as $(\cdot,\cdot)_Q$, $\|\cdot\|_Q$ and $(\cdot,\cdot)_{\partial Q}$, $\|\cdot\|_{\partial Q}$ respectively. To describe the impedance λ we use the spaces $L^\infty{}_{\lambda 0}(\Gamma) = \{\lambda \text{ in } L^\infty(\Gamma): \lambda(x) \geq \lambda_0\}$ and $H^s{}_{\lambda 0}(\Gamma) = \{\lambda \text{ in } H^s(\Gamma): \lambda(x) \geq \lambda_0\}$ with $\lambda_0 > 0$. Note then when $s > 1/2$ and Γ belongs to $C^{1,1}$ the embedding $H^s(\Gamma)$ to $L^\infty(\Gamma)$ is continuous and compact. For describing the incident wave we consider the space $H^{inc}(\Omega_e) = \{u \text{ in } H^1(Q): \Delta u + k^2 u = 0$ as a distribution$\}$.

In order to reduce the problem Eq. 1 – Eq. 2 to the equivalent problem considered in the bounded domain Ω_e we define a Dirichlet-to-Neumann mapping T: $H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ that maps every function g in $H^{1/2}(\Gamma_R)$ to a function $\partial v / \partial n$ where v is a solution of the exterior Dirichlet problem for Helmholtz equation with boundary condition $v|_{\Gamma R} = g$. It is well known that Eq. 1 – Eq. 2 are equivalent to Eq. 1 in Ω_e and impedance boundary condition from Eq. 2 with the following boundary condition:

$$\partial u^s / \partial n = \mathrm{T}\, u^s \quad \text{on } \Gamma_R. \tag{3}$$

For brevity this problem will be referred to as problem 1.

Let u^{inc} belongs to $H^{inc}(\Omega_e)$. We multiply the Eq. 1 by φ^* where φ in $H^1(\Omega_e)$ is a test function, φ^* means the complex conjugate of φ, integrate over Ω_e and apply Green formula. We have

$$\int_{\Omega e} (\mathbf{grad}\, u \cdot \mathbf{grad}\, \varphi^* - k^2 u\, \varphi^*)\, dx = -\int_\Gamma \varphi^*\, \partial u / \partial n\, d\sigma + \int_{\Gamma R} \varphi^*\, \partial u / \partial n\, d\sigma. \tag{4}$$

Taking into account boundary conditions Eq. 2 and Eq. 3 we rewrite Eq. 4 in the form

$$a_0(u, \varphi) - i\, k\, (\lambda\, u, \varphi)_\Gamma = \langle f, \varphi\rangle \quad \text{for any } \varphi \text{ in } H^1(\Omega_e), \tag{5}$$

where $a_0(\cdot,\cdot)$, $(\lambda\,\cdot,\cdot)$ and $\langle f, \cdot\rangle$ are the following sesquiliniear and linear forms:

$$a_0(u, \varphi) := \int_{\Omega e} (\mathbf{grad}\, u \cdot \mathbf{grad}\, \varphi^* - k^2 u\, \varphi^*)\, dx - \int_{\Gamma r} \varphi^*\, \mathrm{T}u\, d\sigma, \qquad (\lambda\, u, \varphi)_\Gamma := \int_\Gamma \lambda\, u\, \varphi^*\, d\sigma,$$
$$\langle f, \varphi\rangle := \int_{\Gamma R} \varphi^* (\partial u^{inc} / \partial n - \mathrm{T}u^{inc})\, d\sigma. \tag{6}$$

The solution of Eq. 5 is called the weak solution of problem 1. Using the properties of forms $a_0(\cdot,\cdot)$, $(\lambda\,\cdot,\cdot)$ and $\langle f, \cdot\rangle$ we can prove the following theorem.

Theorem 1. Under conditions (i) let λ in $L^\infty{}_{\lambda 0}(\Gamma)$ be an arbitrary function where $\lambda_0 > 0$. Then for every incident wave u^{inc} in $H^{inc}(\Omega_e)$ there exist a unique solution u_λ to Eq. 5 that satisfies the following estimate with a constant C_λ that depends on λ and is independent of u^{inc}:

$$\| u_\lambda \|_{1,\Omega e} \leq C_\lambda \|u^{inc}\|_{1,\Omega e}. \tag{7}$$

Assuming that the impedance λ belongs to a nonempty bounded subset K of $L^{\infty}_{\lambda 0}(\Gamma)$ we can prove the following theorem.

Theorem 2. Under conditions (i) let λ in K where K is a nonempty bounded subset of $L^{\infty}_{\lambda 0}(\Gamma)$, $\lambda_0 > 0$. Then for every u^{inc} in $H^{inc}(\Omega_e)$ a unique solution u to Eq. 5 satisfies the following estimate:

$$\|u\|_{1,\,\Omega e} \le C_0\, \|u^{inc}\|_{1,\,\Omega e}, \tag{8}$$

with constant C_0 independent of λ.

Control Problem and Optimality System

Now we are able to formulate our control problem for model Eq. 1, Eq. 2. This problem is to minimize a certain cost functional depending on the state (wave field) u and the unknown function (control), which satisfy the equations of state in the form of Eq. 5 of problem 1. The control is specified by the impedance λ, while the cost functional is either one the following two:

$$I_1(u) = \|u - u^d\|^2_Q = \int_Q |u - u^d|^2\, dx, \qquad I_2(u) = \|u - u^d\|^2 = \int_{\Gamma q} |u - u^d|^2\, d\sigma. \tag{9}$$

Here Q is a subdomain of Ω_e, Γ_q is a continuous cycled curve in Ω_e. When $u^d = u^{inc}$ the functional I_1 (or I_2) is the squared mean square integral norm of the scattered field u^s over Q (or over Γ_q). Assume the following conditions hold:
(j) Γ belongs to $C^{1,\,1}$; $\alpha_0 > 0$; and K is a nonempty convex closed subset of $H_{\lambda 0}{}^s(\Gamma)$, where s > 1/2 and $\lambda_0 > 0$.

Introducing the operator G: $H^1(\Omega_e) \times K \times H^{inc}(\Omega_e) \to H^1(\Omega_e)^*$ by $\langle G(u, \lambda, u^{inc}), \varphi\rangle = a_0(u, \varphi) - i\, k\, (\lambda\, u, \varphi)_\Gamma - \langle f, \varphi\rangle$ for every φ from $H^1(\Omega_e)$, where $H^1(\Omega_e)^*$ is a dual space to $H^1(\Omega_e)$, we rewrite Eq. 5 in the form $G(u, \lambda, u^{inc}) = 0$. Consider the constrained minimization problem

$$J(u, \lambda) := (\alpha_0 / 2)\, I(u) + (\alpha_1 / 2)\, \|\lambda\|^2_{s,\,\Gamma} \to \inf,\ \ G(u, \lambda, u^{inc}) = 0,\ \ (u, \lambda) \text{ in } H^1(\Omega_e) \times K. \tag{10}$$

Here $I = I_k$, k = 1, 2. Proceeding as in [3] we arrive to the following theorem.

Theorem 3. Under conditions (i), (j) let $\alpha_0 > 0$, $\alpha_1 \ge 0$ and let K be a bounded set and u^{inc} in $H^{inc}(\Omega_e)$. Then control problem Eq. 11 has a least one solution (u, λ) in $H^1(\Omega_e) \times K$ for $I = I_k$, k = 1, 2.

The further analysis of problem Eq. 10 consists in using the extremum principle in smooth convex extremum problems [5]. It allows us to derive the necessary extremum conditions and leads to the following theorem.

Theorem 4. Under conditions (i), (j) let (u, λ) in $H^1(\Omega_e) \times K$ be a solution of problem Eq. 10 at $I = I_k$, k = 1, 2. Then there exists a unique Lagrange multiplier p in $H^1(\Omega_e)$ that satisfies the Euler-Lagrange equation

$$a_0(\varphi, p) - i\, k\, (\lambda\, \varphi, p)_\Gamma = -(\alpha_0 / 2)\, \langle I'_u(u), \varphi\rangle^* \quad \text{for any } \varphi \text{ in } H^1(\Omega_e), \tag{11}$$

and the following minimum principle holds:

$$\alpha_1\, (\lambda, \mu - \lambda)_{s,\,\Gamma} - \mathrm{Re}[i\, k\, ((\mu - \lambda)\, u, p)_\Gamma] \ge 0 \quad \text{for any } \mu \text{ in K}. \tag{12}$$

Here I'_u is a Fréchet derivative of the functional I.

The weak formulation of problem 1, Eq. 5, Eq. 11, which is interpreted as an adjoint problem for the adjoint state p in $H^1(\Omega_e)$, and variational inequality Eq. 12 form an optimality system of problem Eq. 10. This system describes the necessary extremum conditions of Eq. 10.

Numerical Algorithm for Control Problem

In order to find the numerical solution of the optimality system we suggest the following algorithm based on the idea of simple iteration. Assume that for any step of the algorithm we know some approximation λ_n. Using the λ_n we find λ_{n+1} by sequential solving the following problems

$$a_0(u_n,\ \varphi) - i\,k\,(\lambda_n u_n,\ \varphi)_\Gamma = \langle f,\ \varphi\rangle \quad \text{for any } \varphi \text{ in } H^1(\Omega_e), \tag{13}$$

$$a_0(\varphi, p_n) - i\,k\,(\lambda_n \varphi, p_n)_\Gamma = -(\alpha_0 / 2)\langle I'_u(u_n), \varphi\rangle^* \quad \text{for any } \varphi \text{ in } H^1(\Omega_e), \tag{14}$$

$$\alpha_1\,(\lambda_{n+1}, \mu - \lambda_{n+1})_{s,\Gamma} - \mathrm{Re}[i\,k\,((\mu - \lambda_{n+1})\,u_n, p_n)_\Gamma] \geq 0 \quad \text{for any } \mu \text{ in K}. \tag{15}$$

Thus, each step of the algorithm consists of finding the solutions u_n and p_n of Eq. 13 and Eq. 14, which are the weak formulations of some external boundary value problems. Using the boundary element approach we can reduce these problems to boundary integral equations. Boundary element method gives direct numerical values of $u_n|_\Gamma$ and $p_n|_\Gamma$ which are used in Eq. 15 and the way to find u_n and p_n in any subdomain of Ω_e.

Summary

We have analyzed the direct scattering impedance problem and the inverse extremal problem of choosing the surface impedance. The solvability of direct and inverse problems is derived. The optimality system constructed. Numerical algorithm of solving the control problem based on optimality system and boundary element method is proposed. The results of numerical simulations will be published in separate authors' work.

Acknowledgements

This work was financially supported by the Russian Science Foundation (project no. 14-11-00079).

References

[1] J.B. Pendry, D. Schurig and D.R. Smith: Science Vol. 312 (2006), p. 1780

[2] Min Yan, Zhichao Ruan, Min Qiu: Optics Express Vol. 15 (2007), p. 17772

[3] Yu. I. Bobrovnitskii: Acoustical Physics Vol. 54 (2008), p. 758

[4] Yu. I. Bobrovnitskii: Doklady Physics Vol. 57 (2012), p. 18

[5] G.V. Alekseev: Doklady Physics Vol. 58 (2013), p. 147.

[6] G.V. Alekseev: Computational Mathematics and Math. Physics Vol. 53 (2013), p. 1853

[7] G.V. Alekseev: Doklady Physics Vol. 58 (2013), p. 482

[8] G.V. Alekseev, A.V. Lobanov: J. Applied and Industrial Mathematics Vol. 7 (2013), p. 302

[9] G.V. Alekseev, V.A. Levin: Doklady Physics Vol. 59 (2014), p. 89

[10] G.V. Alekseev: Applicable Analysis. Vol. 93 (2014), p. 254

[11] Ju.Yu. Fershalov, T.V. Sazonov: Advanced Materials Research Vol. 915-916 (2014), p. 345

[12] M.Yu. Fershalov, A.Yu. Fershalov, Ju.Yu. Fershalov: Advanced Materials Research Vol. 915-916 (2014), p. 341

[13] Yu.Ya. Fershalov: Russian Aeronautics Vol. 55 (2014), p. 424