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Formation of the Residual Stress Field under Local Thermal Actions

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Abstract—The one-dimensional process of material deformation due to local heating and subsequent cooling is analyzed in the framework of the classical theory of elastoplastic deformations. The problem of formation of residual stresses in a thin plate made of an elastoplastic material under a given thermal action is solved. The graphs of fields of residual stresses and displacements are constructed.

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1. INTRODUCTION

The field of residual stresses is formed in the process of elastoplastic deformation with subsequent unloading of the material deformed in this way. It is known that residual stresses may arise because of local thermal actions, for example, near the welding joints [1]. Obviously, in this case the process of elastoplastic deformation is initiated by fast heating of the material along the welding line. There are known engineering methods for calculating the residual stresses in the region of thermal influence of the welding joint [2]. Here we present the exact solution of the problem of formation of residual stresses under the assumption that the connection between the processes of heat conduction and deformation under the conditions of intensive thermal action can be neglected, i.e., the calculations can be performed in the framework of the theory of thermal stresses [3, 4] with the yield point dependence on the temperature taken into account.

2. BASIC MODEL DEPENDENCIES

We assume that, till the time $t = 0$, the plate is in free state at room temperature T_0. We also assume that the elastoplastic material of the plate obeys the Prandtl–Reiss-type model [5, 6], where the strains e_{ij} are assumed to be small and are composed of elastic $e^{\rm e}_{ij}$ and plastic $e^{\rm p}_{ij}$ residual strains

$$e_{ij} = e^{\rm e}_{ij} + e^{\rm p}_{ij} = 0.5(u_{i,j} + u_{j,i}). \tag{2.1}$$

The level and distribution of elastic strains and the temperature over the plate determine the stresses in the plate which obey the Duhamel–Neumann law

$$\sigma_{ij} = (\lambda e^{\rm e}_{kk} + 3\alpha K\theta)\delta_{ij} + 2\mu e^{\rm e}_{ij}, \quad \theta = T(r,t) - T_0. \tag{2.2}$$

Here λ and μ are the Lamé parameters, K is the modulus of uniform compression of the material ($3K = 3\lambda + 2\mu$), α is the coefficient of linear temperature expansion of the material, and $T(r,t)$ is the

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current temperature. The plastic flow process in the plate material begins at the time instant when the plasticity condition in the Tresca form [7]

$$f(\sigma_{ij}) = \max|\sigma_i - \sigma_j| - 2k(\theta) = 0 \tag{2.3}$$

is satisfied, where the σ_i are the principal stresses and $k(\theta)$ is the yield point of the material at a given temperature. In our further calculations, we consider the simplest linear dependence $k(\theta) = k_0 - \beta\theta$, where k_0 is the yield point of the material at room temperature and β is the material thermophysical constant determining the degree of the yield strength decrease with increasing temperature. Under the conditions of the accepted von Mises maximum principle [7], the surface (2.3) becomes a plastic potential which implies the associated plastic flow law

$$\varepsilon^{\mathrm{p}}_{ij} = \xi\frac{\partial f}{\partial\sigma_{ij}}, \quad \xi = \sqrt{\varepsilon^{\mathrm{p}}_{kl}\varepsilon^{\mathrm{lk}}_{ij}}\left(\frac{\partial f}{\partial\sigma_{mn}}\frac{\partial f}{\partial\sigma_{nm}}\right)^{-1/2}. \tag{2.4}$$

If we supplement relations (2.1)–(2.4) with local consequences of the conservation laws (equation of motion and equation of the internal energy balance) and postulate the heat conduction law, for example, in the Fourier form, then we obtain a closed mathematical model of deformation.

3. STATEMENT OF THE PROBLEM. ELASTIC DEFORMATIONS

We assume that an infinite plate is heated so that the temperature on its line which is a circle of radius R increases proportionally to time. This action increases the stresses and strains in the plate material. We assume that the plate thickness is sufficiently small and the plate is under the conditions of plane stress state so that in the cylindrical coordinates r, φ, z the normal stresses σ_{zz} on the sites orthogonal to the plate are zero,

$$\begin{gathered}\sigma_{zz} = (\lambda + 2\mu)e_{zz} + \lambda(e_{rr} + e_{\varphi\varphi}) - 3\alpha K\theta = 0,\\ e_{rr} = e^{\mathrm{e}}_{rr} = u_{r,r}, \quad e_{zz} = e^{\mathrm{e}}_{zz} = u_{z,z}, \quad e_{\varphi\varphi} = e^{\mathrm{e}}_{\varphi\varphi} = \frac{u_r}{r}.\end{gathered} \tag{3.1}$$

According to (2.2), for the other nonzero components of the stress tensor we then have

$$\begin{gathered}\sigma_{rr} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{r,r} + \frac{2\lambda\mu}{\lambda+2\mu}\frac{u_r}{r} - \frac{6\alpha\mu K}{\lambda+2\mu}\theta,\\ \sigma_{\varphi\varphi} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{u_r}{r} + \frac{2\lambda\mu}{\lambda+2\mu}u_{r,r} - \frac{6\alpha\mu K}{\lambda+2\mu}\theta.\end{gathered} \tag{3.2}$$

By substituting the stress components from (3.2) into the equilibrium equations written in cylindrical coordinates, we obtain the following equation for the only nonzero component of the displacement vector:

$$u_{r,rr} + \frac{u_{r,r}}{r} - \frac{u_r}{r^2} = \frac{3\alpha K}{2(\lambda+\mu)}\theta_{,r}. \tag{3.3}$$

If in the energy balance equation (a local version of the energy conservation law), we neglect the heat source due to deformation (connection between the processes of deformation and heat transfer) and accept the simplest linear dependence of the heat flow on the temperature gradient, then for the nonstationary distribution of temperature over the plate we obtain the classical hear conduction equation [8, 9]. Its solution is known in the case under study and for the accepted initial and boundary conditions [9]. We can use this solution (we do not write it out because of its cumbersomeness) or use well-known software that also permits obtaining the temperature distribution over the plate heated from time $t = 0$ along the line $r = R$ till the temperature T_k on this line. To obtain the simplest solution, we assume that the temperature on the line $r = R$ increases proportionally to time from the room temperature T_0 to the final heating temperature T_k. The latter can be arbitrary including the melting temperature for the plate material. At temperatures so high, it is expedient to take into account the dependence of the heat conductivity coefficient on the temperature gradient and the temperature variation rate. But, to obtain an exact solution, we neglect such dependencies and assume that the heat conductivity coefficient is constant. At a higher temperature level, the temperature stresses may cause solid phase transitions that significantly change the structure of the plate material. The latter lead to irreversible deformations which we further take into account as plastic strains and to variations in the Lamé parameters λ and μ in the thermal influence region of (the phase transition zone). For our purposes (i.e., to obtain the exact solution), we do not consider such variations, i.e., we assume that λ and μ are constant.

From the known distribution $\theta(r,t)$, we can find the displacements by integrating Eq. (3.3) separately in the regions $r < R$ and $r > R$. The calculated displacement field is also used to obtain the stress distribution. For the domain $r < R$, we obtain

$$u_r = \frac{b}{\mu r}\psi(r,t) + c_1 r, \quad \sigma_{rr} = -\frac{2b}{r^2}\psi(r,t) + qc_1, \quad \sigma_{\varphi\varphi} = \frac{2b}{r^2}[\psi(r,t) - r^2\theta] + qc_1,$$
$$b = \frac{3\alpha\mu K}{2(\lambda+\mu)}, \quad q = \frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu}, \quad \psi(r,t) = \int\limits_0^r \theta(\rho,t)\rho\,d\rho. \tag{3.4}$$

In the domain $r > R$, we obtain

$$u_r = \frac{b}{\mu r}\phi(r,t) + \frac{c_2}{r}, \quad \sigma_{rr} = -\frac{2b}{r^2}\phi(r,t) - 2\mu\frac{c_2}{r^2}, \quad \sigma_{\varphi\varphi} = -\sigma_{rr} - 2b\theta, \quad \phi(r,t) = \int\limits_R^r \theta(\rho,t)\rho\,d\rho. \tag{3.5}$$

By satisfying the continuity conditions for the displacements and stresses on the boundary $r = R$, we obtain the constants of integration c_1 and c_2:

$$c_1 = 0, \quad c_2 = b\psi(R,t). \tag{3.6}$$

The dependences (3.4)–(3.6) solve our problem, but they have a restriction related to the output of stress states (3.4) and (3.5) to the loading surface, which starts the plastic flow in the plate material. This fact is related to the restrictions on the heating rate; namely, it must be sufficiently large to cause an appropriate level of temperature stresses. The plastic flow region evolves from the line $r = R$ on both sides and, at a certain current moment of heating, occupies the domain between the lines $r = r_1$ and $r = r_2$ ($r_1 < R < r_2$). Outside this domain, the plate deformation is still reversible and the quasistatic parameters of such stress-strain states are still determined by (3.4)–(3.5) with the only difference that the constants of integration are determined by the values different from (3.6),

$$c_1(r_1) = \frac{2}{q}\left(-\frac{b}{r_1^2}\psi(r_1,t) + b\theta_1 - k_1\right), \quad c_2(r_2) = \frac{r_2^2}{\mu}(b\theta_2 - k_2),$$
$$\theta_n = \theta(r_n), \quad k_n = k(\theta_n), \quad n = 1,2. \tag{3.7}$$

In the evolving plastic flow region, the stress states correspond to points of the loading surface (2.3). In the case under study, these are points of the Tresca prism lying in the plane $\sigma_{zz} = 0$ of the principal stress space on the line

$$\sigma_{\varphi\varphi} = 2k(\theta). \tag{3.8}$$

By integrating the equilibrium equation under this condition, we obtain, in the entire flow region,

$$\sigma_{rr} = -\frac{2}{r}\int\limits_{r_1}^r k(\theta(\rho,t))\,d\rho + \frac{c_3}{r}. \tag{3.9}$$

The reversible strains at any time and any point of the plastic flow region can now be calculated by formulas (3.8) and (3.9) expressing them in terms of (2.2) written in cylindrical coordinates,

$$
\begin{gathered}
e^{\mathrm{e}}_{\phi\phi} = 2s[b\theta - k(\theta)] - \frac{g}{r}\left[\frac{c_3}{2} - \int\limits_{r_1}^{r} k(\theta(\rho,t))\,d\rho\right], \\
e^{\mathrm{e}}_{rr} = 2s\left\{b\theta + \frac{1}{r}\left[\frac{c_3}{2} - \int\limits_{r_1}^{r} k(\theta(\rho,t))\,d\rho\right]\right\} + gk(\theta), \\
s = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}, \quad g = \frac{\lambda}{\mu(3\lambda+2\mu)}.
\end{gathered}
\tag{3.10}
$$

The associated plastic flow law (2.4) implies the relation

$$
e^{\mathrm{p}}_{rr} = u_{r,r} - e^{\mathrm{e}}_{rr} = 0. \tag{3.11}
$$

By substituting (3.10) into (3.11), we obtain the equation for the displacements in the flow region,

$$
u_{r,r} = 2s\left\{b\theta + \frac{1}{r}\left[\frac{c_3}{2} - \int\limits_{r_1}^{r} k(\theta(\rho,t))\,d\rho\right]\right\} + gk(\theta). \tag{3.12}
$$

We integrate this equation and obtain

$$
u_r = 2s\left[b\int\limits_{r_1}^{r}\theta(\rho,t)\,d\rho + \frac{c_3}{2}\ln\frac{r}{r_1} - \int\limits_{r_1}^{r}\frac{1}{\rho}\int\limits_{r_1}^{\rho} k(\theta(\chi,t))\,d\chi\,d\rho\right] + g\int\limits_{r_1}^{r} k(\theta(\rho,t))\,d\rho + c_4. \tag{3.13}
$$

The unknown constants of integration c_3 and c_4 and the dimensions of the irreversible deformation region ($r_1 < r < r_2$) can be found from the conditions that the displacement and stress fields are continuous on the elastoplastic boundaries (r_1, r_2). After simple transformations, we obtain the relations

$$
\begin{gathered}
c_3(r_1) = 2r_1\left[b\theta_1 - \frac{2b}{r_1^2}\psi(r_1,t) - k_1\right], \\
c_4(r_1) = \frac{2br_1}{q}\left(\theta_1 - \frac{k_1}{b}\right) + \left(\frac{1}{\mu} - \frac{2}{q}\right)\frac{b}{r_1}\psi(r_1,t),
\end{gathered}
\tag{3.14}
$$

and the system for determining the boundaries of the plastic flow region

$$
\begin{cases}
b\int\limits_{r_1}^{r_2}\theta(\rho,t)\,d\rho - \int\limits_{r_1}^{r_2}\frac{1}{\rho}\int\limits_{r_1}^{\rho} k(\theta(\chi,t))\,d\chi\,d\rho - \int\limits_{r_1}^{r_2} k(\theta(\rho,t))\,d\rho \\
\qquad + \frac{2b}{r_1}\psi(r_1,t) + \left(\frac{1}{2}\ln\frac{r_2}{r_1} + 1\right)\left\{2r_1\left[b\theta_1 - \frac{2b}{r_1^2}\psi(r_1,t) - k_1\right]\right\} = 0, \\
b\theta_2 r_2 - r_2 k_2 = \int\limits_{r_1}^{r_2} k(\theta(\rho,t))\,d\rho + \frac{2b}{r_1}\psi(r_1,t) + 2r_1k_1 - b\theta_1 r_1.
\end{cases}
\tag{3.15}
$$

The displacement and stress field distribution at the final moment of heating $t = \tilde{t}$ are shown in Figs. 1 and 2, respectively. The calculations were performed for the following constants of the material: $\alpha \cdot (T_k - T_0) = 6.2 \times 10^{-3}$, $\lambda/\mu = 1.5$, $k_0/\mu = 1.1 \times 10^{-3}$, and $\beta(T_k - T_0)/\mu = 1.09 \times 10^{-3}$.

4. COOLING OF THE PLATE

At a time $\tilde{t}$, we begin to cool the plate. In what follows, the tilde over a symbol denotes the functions and constants calculated at time $\tilde{t}$. Then, in the reversible deformation regions, the displacements and stresses are determined by (3.4)–(3.5) up to the constants of integration.

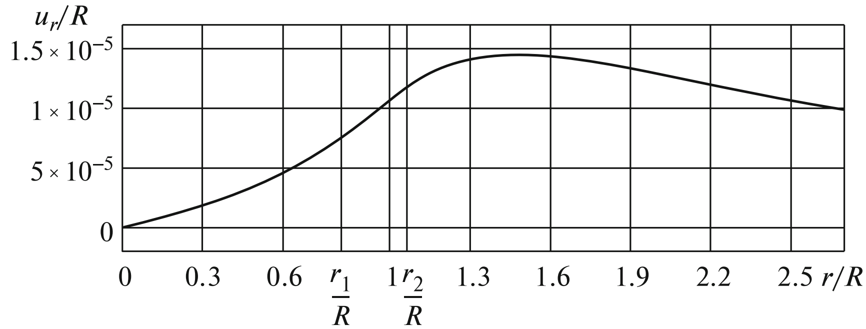

Fig. 1.

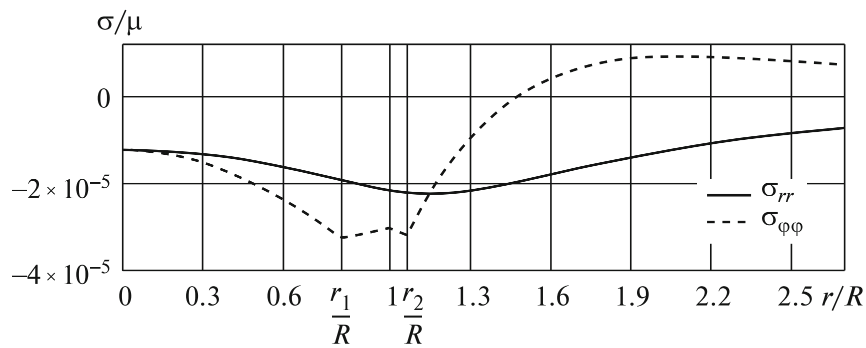

Fig. 2.

In the region with accumulated irreversible deformations $\tilde{r}_1 < r < \tilde{r}_2$, the plastic deformations do not change under unloading and are determined by the dependences

$$e^{\mathrm{p}}_{\varphi\varphi} = -\frac{2bs}{r}\int\limits_{\tilde{r}_1}^{r} \tilde{\theta}(\rho)\,d\rho + \frac{2s}{r}\int\limits_{\tilde{r}_1}^{r} k(\tilde{\theta}(\rho))\ln\rho\,d\rho + \left(\frac{s}{r}\ln\frac{r}{\tilde{r}_1} + \frac{g}{2r}\right)\tilde{c}_3$$
$$+\frac{2s}{r}\ln r\int\limits_{\tilde{r}_1}^{r} k(\tilde{\theta}(\rho))\,d\rho + \frac{\tilde{c}_4}{r} - 2bs\tilde{\theta} + 2sk(\tilde{\theta}), \quad e^{\mathrm{p}}_{rr} = 0. \tag{4.1}$$

The elastic deformations are calculated by (2.1),

$$e^{\mathrm{e}}_{rr} = u_{r,r}, \quad e^{\mathrm{e}}_{\varphi\varphi} = \frac{u}{r} - e^{\mathrm{p}}_{\varphi\varphi}. \tag{4.2}$$

We use (4.2) to write out the Duhamel–Neumann law (2.2) in the form

$$\begin{aligned}\sigma_{rr} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{r,r} + \frac{2\lambda\mu}{\lambda+2\mu}\left(\frac{u_r}{r} - e^{\mathrm{p}}_{\varphi\varphi}\right) - \frac{6\alpha\mu K}{\lambda+2\mu}\theta,\\ \sigma_{\varphi\varphi} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{u_r}{r} - e^{\mathrm{p}}_{\varphi\varphi}\right) + \frac{2\lambda\mu}{\lambda+2\mu}u_{r,r} - \frac{6\alpha\mu K}{\lambda+2\mu}\theta.\end{aligned} \tag{4.3}$$

By substituting the stress tensor components (4.3) into the equilibrium equation and by replacing the irreversible deformation components by (4.1), we obtain the following equation for the radial stress component:

$$u_{r,rr} + \frac{u_{r,r}}{r} - \frac{u_r}{r^2} = -\frac{1}{qs}\frac{e^{\mathrm{p}}_{\varphi\varphi}}{r} + \frac{g}{2s}e^{\mathrm{p}}_{\varphi\varphi,r} + \frac{b}{\mu}\theta_{,r}. \tag{4.4}$$

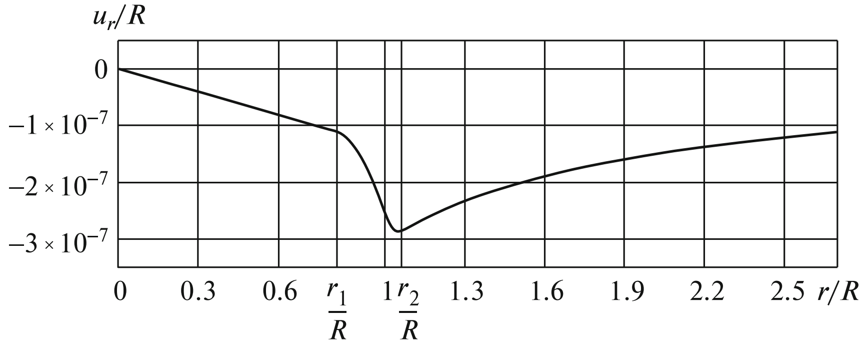

Fig. 3.

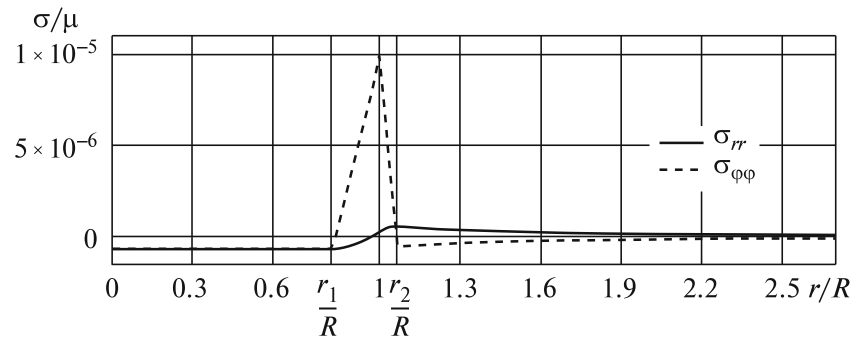

Fig. 4.

The solution of Eq. (4.4) is the function

$$u_r = \frac{b}{\mu r}\int\limits_{\tilde r_1}^{r}\theta(\rho,t)\rho\,d\rho - \frac{r}{2qs}\int\limits_{\tilde r_1}^{r}\frac{e^{\mathrm{p}}_{\varphi\varphi}(\rho)}{\rho} + \frac{1}{s\mu r}\int\limits_{\tilde r_1}^{r} e^{\mathrm{p}}_{\varphi\varphi}(\rho)\rho\,d\rho + \frac{(r^2-\tilde r_1^2)c_5}{2qsr} + \frac{c_6}{r}, \tag{4.5}$$

which allows us to calculate the stress (4.3) in the region with accumulated irreversible deformations

$$\sigma_{rr} = -\frac{2b}{r^2}\int\limits_{\tilde r_1}^{r}\theta(\rho,t)\,d\rho - \frac{1}{2s}\left[\int\limits_{\tilde r_1}^{r}\frac{e^{\mathrm{p}}_{\varphi\varphi}(\rho)}{\rho}d\rho + \frac{1}{r^2}\int\limits_{\tilde r_1}^{r} e^{\mathrm{p}}_{\varphi\varphi}(\rho)\rho\,d\rho\right] + \mu\left(2 + \frac{r^2-\tilde r_1^2}{2r^2sq}\right)c_5 - \frac{2\mu}{r^2}c_6,$$

$$\sigma_{\phi\phi} = \frac{2b}{r^2}\int\limits_{\tilde r_1}^{r}\theta(\rho,t)\,d\rho - \frac{1}{2s}\left[\int\limits_{\tilde r_1}^{r}\frac{e^{\mathrm{p}}_{\varphi\varphi}(\rho)}{\rho}d\rho + \frac{1}{r^2}\int\limits_{\tilde r_1}^{r} e^{\mathrm{p}}_{\varphi\varphi}(\rho)\rho\,d\rho\right] \tag{4.6}$$

$$+ \mu\left(2 + \frac{r^2-\tilde r_1^2}{2r^2sq}\right)c_5 - \frac{e^{\mathrm{p}}_{\varphi\varphi}}{s} - 2b\theta + \frac{2\mu}{r^2}c_6.$$

We again use the conditions that the displacement and stress fields are continuous on the elastoplastic boundaries $(\tilde r_1, \tilde r_2)$ to obtain the unknown constants of integration. After simple transformations, we obtain

$$c_1(\tilde r_1,\tilde r_2) = \frac{1}{sq}\int\limits_{\tilde r_1}^{\tilde r_2}\frac{e^{\mathrm{p}}_{\varphi\varphi}(\rho)}{\rho}d\rho, \quad c_2(\tilde r_1,\tilde r_2) = \frac{b}{\mu}\psi(\tilde r_1,t) + \frac{1}{4\mu s}\int\limits_{\tilde r_1}^{\tilde r_2} e^{\mathrm{p}}_{\varphi\varphi}(\rho)\,d\rho,$$

$$c_5(\tilde r_1,\tilde r_2) = \int\limits_{\tilde r_1}^{\tilde r_2}\frac{e^{\mathrm{p}}_{\varphi\varphi}(\rho)}{\rho}d\rho, \quad c_6(\tilde r_1,\tilde r_2) = \frac{b}{\mu}\psi(\tilde r_1,t) + c_1(\tilde r_1,\tilde r_2)\tilde r_1^2. \tag{4.7}$$

Figures 3 and 4 illustrate the displacement and residual stress fields at the time when the body attains its initial temperature.

5. CONCLUSION

The constructed solution permits predicting the level of residual stresses near the line $r = R$, where the temperature action is localized, and the dimensions of the irreversible deformation region. If this solution is applied to the welding processes, then one should note the following: the dimensions of the irreversible deformation region can only approximately be identified with the thermal influence region, just as the level of calculated residual stresses. More precise calculations are possible only if we use the experimentally verified dependence of the yield point on the temperature and take into account the variations in the elastic constants and the yield strength due to the solid phase transitions in the material structure. It is the decrease in the values of the Lamé parameters that leads to a decrease in the strength properties of the material in the region of the welding joint. This problem has not yet been solved exactly.

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