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ON SOME BOUNDS FOR REAL PARTS OF THE CRITICAL POINTS OF POLYNOMIALS

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Abstract. Using recent results on companion matrices and some bounds for eigenvalues we get inequalities for real parts of the critical points of polynomials.

1. Introduction

Relations between zeros and critical points of polynomials are due to Gauss-Lucas theorem and have rich history. In many cases proofs of them are based on matrix theory (see, for example [1], [3]–[5], and references in them). In recent paper [3] W. S. Cheung and T. W. Ng introduced a new derivative companion matrix and used it to prove Bruin and Sharma's conjecture, and some results on the majorization of the critical points of a polynomial by its zeros. Also in the article [1] Mohammad Adm., F. Kittaneh obtained several inequalities of this type using the same matrix. The aim of this short paper is to supplement and improve some results from the paper [1] for real parts of critical points applying eigenvalues theorem to a matrix which is similar to the matrix mentioned above. Using multiplication of the argument by unimodal constant this kind of theorems can be used to localize the critical points of a polynomial. Our approach also allows to study stability of derivative of a given polynomial by varying its zeros.

Let P be a polynomial of degree $n \geqslant 3$, with complex coefficients, and let $z_1, z_2, \dots,$ z_n be the zeros of P. Let

$$D = \begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_{n-1} \end{pmatrix},$$

I and J be the identity matrix of order $(n-1)$ and the $(n-1)\times(n-1)$ matrix with all entries equal to 1, respectively. In [3] W. S. Cheung and T. W. Ng proved the following statement.

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STATEMENT 1. *The set of all eigenvalues of the* $(n-1)\times(n-1)$ *matrix*

$$D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J$$

is the same as the set of all critical points of the polynomial P.

REMARK 1. In [3] it was also shown that the symmetric matrices $C(P')=(I-\alpha J)D(I-\alpha J)+(z_n/n)J$ and $D(I-J/n)+(z_n/n)J$ are similar and therefore have the same set of eigenvalues, if $\alpha=1/(n-\sqrt{n})$.

2. Preliminary results

Let $M_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices. For $A\in M_n(\mathbb{C})$, the eigenvalues of A are denoted by $\lambda_1(A),\lambda_2(A),\dots,\lambda_n(A)$, and $A^*=\overline{A}^T$. If A is Hermitian, then the eigenvalues of A are arranged in such a way that $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\dots\geqslant\lambda_n(A)$. The following lemmas can be found in the books [2, 6].

LEMMA 1. *Let* $A\in M_n(\mathbb{C})$ *with real part* $\mathrm{Re}A=\dfrac{A+A^*}{2}$. *Then* $\lambda_n(\mathrm{Re}A)\leqslant\mathrm{Re}\lambda_j(A)\leqslant\lambda_1(\mathrm{Re}A)$ *for* $j=1,2,\dots,n$.

LEMMA 2. *Let* $A_1,A_2,\dots,A_m\in M_n(\mathbb{C})$ *be Hermitian. Then*

$$\lambda_j(A_1)+\sum_{i=2}^{m}\lambda_n(A_i)\leqslant\lambda_j\left(\sum_{i=1}^{m}A_i\right)\leqslant\lambda_j(A_1)+\sum_{i=2}^{m}\lambda_1(A_i)\ \text{for}\ j=1,2,\dots,n.$$

In particular,

$$\lambda_1\left(\sum_{i=1}^{m}A_i\right)\leqslant\sum_{i=1}^{m}\lambda_1(A_i)$$

and

$$\sum_{i=1}^{m}\lambda_n(A_i)\leqslant\lambda_n\left(\sum_{i=1}^{m}A_i\right).$$

LEMMA 3. *Let* $A\in M_n(\mathbb{C})$ *with eigenvalues arranged in such way that* $\mathrm{Re}\lambda_1(A)\geqslant\mathrm{Re}\lambda_2(A)\geqslant\dots\geqslant\mathrm{Re}\lambda_n(A)$. *Then*

$$\sum_{j=1}^{k}\mathrm{Re}\lambda_j(A)\leqslant\sum_{j=1}^{k}\lambda_j(\mathrm{Re}A),\ \text{for}\ k=1,2,\dots,n-1$$

and

$$\sum_{j=1}^{n}\mathrm{Re}\lambda_j(A)=\sum_{j=1}^{n}\lambda_j(\mathrm{Re}A).$$

LEMMA 4. *Let $A_1, A_2, \ldots, A_m \in M_n(\mathbb{C})$ be Hermitian. Then*

$$\sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{m} A_i \right) \leqslant \sum_{j=1}^{k} \left(\sum_{i=1}^{m} \lambda_j(A_i) \right) \quad \textit{for} \;\; k = 1, 2, \ldots, n-1$$

and

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} A_i \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \lambda_j(A_i) \right).$$

3. Main results

To simplify statements of theorems we introduce the following notation

$$\Lambda(\alpha, z_1, \ldots, z_n) = \alpha^2 \sum_{i=1}^{n-1} \operatorname{Re} z_i + \frac{\operatorname{Re} z_n}{n}.$$

THEOREM 1. *Let $z_1, z_2, \ldots, z_n$ be the zeros of a polynomial P of degree $n \geqslant 3$ and $w_1, w_2, \ldots, w_{n-1}$ be the critical points of P. Then for $j = 1, 2, \ldots, n-1$, we have*

$$\min_{1 \leqslant j \leqslant n-1} \{(1-2\alpha)\operatorname{Re} z_j\} - \alpha\sqrt{n-2} \sum_{i=1}^{n-1} |\operatorname{Re} z_i| + \Lambda(\alpha, z_1, \ldots, z_n) \leqslant \operatorname{Re} w_j$$

$$\leqslant \max_{1 \leqslant j \leqslant n-1} \{(1-2\alpha)\operatorname{Re} z_j\} + \alpha\sqrt{n-2} \sum_{i=1}^{n-1} |\operatorname{Re} z_i|, \; \textit{if} \; \Lambda(\alpha, z_1, \ldots, z_n) < 0;$$

$$\min_{1 \leqslant j \leqslant n-1} \{(1-2\alpha)\operatorname{Re} z_j\} - \alpha\sqrt{n-2} \sum_{i=1}^{n-1} |\operatorname{Re} z_i| \leqslant \operatorname{Re} w_j \leqslant \max_{1 \leqslant j \leqslant n-1} \{(1-2\alpha)\operatorname{Re} z_j\}$$

$$+\alpha\sqrt{n-2} \sum_{i=1}^{n-1} |\operatorname{Re} z_i| + \Lambda(\alpha, z_1, \ldots, z_n), \; \textit{if} \; \Lambda(\alpha, z_1, \ldots, z_n) \geqslant 0.$$

Proof. It follows from the definition of $C(P')$ that

$$\begin{aligned}
\operatorname{Re} C(P') &= (I - \alpha J)\operatorname{Re} D(I - \alpha J) + \frac{\operatorname{Re} z_n}{n} J \\
&= \operatorname{Re} D - \alpha \cdot \begin{pmatrix} 2\operatorname{Re} z_1 & \operatorname{Re} z_1 + \operatorname{Re} z_2 & \ldots & \operatorname{Re} z_1 + \operatorname{Re} z_{n-1} \\ \operatorname{Re} z_2 + \operatorname{Re} z_1 & 2\operatorname{Re} z_2 & \ldots & \operatorname{Re} z_2 + \operatorname{Re} z_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Re} z_{n-1} + \operatorname{Re} z_1 & \operatorname{Re} z_{n-1} + \operatorname{Re} z_2 & \ldots & 2\operatorname{Re} z_{n-1} \end{pmatrix} \\
&\quad + \alpha^2 \left(\sum_{i=1}^{n-1} \operatorname{Re} z_i \right) J + \frac{\operatorname{Re} z_n}{n} J.
\end{aligned}$$

Thus, $\mathrm{Re}\,C(P') = A_1 + A_2 + A_3$, where

$$A_1 = (1-2\alpha)\begin{pmatrix} \mathrm{Re}\,z_1 & 0 & \dots & 0 \\ 0 & \mathrm{Re}\,z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{Re}\,z_{n-1} \end{pmatrix},$$

$$A_2 = -\alpha\begin{pmatrix} 0 & \mathrm{Re}\,z_1 + \mathrm{Re}\,z_2 & \dots & \mathrm{Re}\,z_1 + \mathrm{Re}\,z_{n-1} \\ \mathrm{Re}\,z_2 + \mathrm{Re}\,z_1 & 0 & \dots & \mathrm{Re}\,z_2 + \mathrm{Re}\,z_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}\,z_{n-1} + \mathrm{Re}\,z_1 & \mathrm{Re}\,z_{n-1} + \mathrm{Re}\,z_2 & \dots & 0 \end{pmatrix},$$

$$A_3 = \Lambda(\alpha, z_1, ..., z_n)\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Now, A_2 can be written as $A_2 = B_1 + B_2 + \ldots + B_{n-1}$, where

$$B_1 = -\alpha\begin{pmatrix} 0 & \mathrm{Re}\,z_1 & \mathrm{Re}\,z_1 & \dots & \mathrm{Re}\,z_1 \\ \mathrm{Re}\,z_1 & 0 & 0 & \dots & 0 \\ \mathrm{Re}\,z_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}\,z_1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$B_2 = -\alpha\begin{pmatrix} 0 & \mathrm{Re}\,z_2 & 0 & \dots & 0 \\ \mathrm{Re}\,z_2 & 0 & \mathrm{Re}\,z_2 & \dots & \mathrm{Re}\,z_2 \\ 0 & \mathrm{Re}\,z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \mathrm{Re}\,z_2 & 0 & \dots & 0 \end{pmatrix}, \dots$$

$$B_{n-1} = -\alpha\begin{pmatrix} 0 & 0 & 0 & \dots & \mathrm{Re}\,z_{n-1} \\ 0 & 0 & 0 & \dots & \mathrm{Re}\,z_{n-1} \\ 0 & 0 & 0 & \dots & \mathrm{Re}\,z_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}\,z_{n-1} & \mathrm{Re}\,z_{n-1} & \mathrm{Re}\,z_{n-1} & \dots & 0 \end{pmatrix},$$

and all of these matrices are Hermitian.

After some simple computations, we have

$$\sigma(A_1) = \{(1-2\alpha)\mathrm{Re}\,z_1, (1-2\alpha)\mathrm{Re}\,z_2, \dots, (1-2\alpha)\mathrm{Re}\,z_{n-1}\},$$

$$\sigma(B_j) = \left\{-\alpha \mathrm{Re}\,z_j\sqrt{n-2}, \alpha\mathrm{Re}\,z_j\sqrt{n-2}, 0, \dots, 0\right\} \text{ for } j = 1, 2, \dots, n-1,$$

where 0 is of multiplicity $n-3$, and

$$\sigma(A_3) = \{(n-1)\Lambda(\alpha, z_1, ..., z_n), 0, \dots, 0\},$$

where 0 is of multiplicity $n-2$. Here $\sigma(A)$ denotes the spectrum (or set of all eigenvalues) of A.

Applying Lemmas 1 and 2 to $C(P')$ and to the Hermitian matrices $A_1, B_1, B_2, \dots, B_{n-1}, A_3$, we obtain

$$\begin{aligned}
&\lambda_n(A_1)+\lambda_n(B_1)+\lambda_n(B_2)+\ldots+\lambda_n(B_{n-1})+\lambda_n(A_3)\\
&\leqslant \lambda_n(\mathrm{Re}C(P')) \leqslant \mathrm{Re}\, w_j \leqslant \lambda_1(\mathrm{Re}C(P'))\\
&\leqslant \lambda_1(A_1)+\lambda_1(B_1)+\lambda_1(B_2)+\ldots+\lambda_1(B_{n-1})+\lambda_1(A_3) \text{ for } j=1,2,\dots,n-1.
\end{aligned}$$

It is sufficient to apply statement 1 and remark 1. The theorem is proved. □

THEOREM 2. *Let $z_1, z_2, \dots, z_n$ be the zeros of a polynomial f of degree $n \geqslant 4$ and $w_1, w_2, \dots, w_{n-1}$ be the critical points of f arranged in such way that $Re\, z_1 \geqslant Re\, z_2 \geqslant \dots \geqslant Re\, z_n$ and $Re\, w_1 \geqslant Re\, w_2 \geqslant \dots \geqslant Re\, w_{n-1}$. Then for $k = 1, 2, \dots, n-2$, we have*

$$\sum_{j=1}^{k} \mathrm{Re}\, w_j \leqslant (1-2\alpha)\sum_{j=1}^{k} \mathrm{Re}\, z_j + \alpha\sqrt{n-2}\sum_{j=1}^{n-1} |\mathrm{Re}\, z_j| + (n-1)\Lambda(\alpha, z_1, ..., z_n),$$

$$\textit{if } \Lambda(\alpha, z_1, ..., z_n) \geqslant 0;$$

$$\sum_{j=1}^{k} \mathrm{Re}\, w_j \leqslant (1-2\alpha)\sum_{j=1}^{k} \mathrm{Re}\, z_j + \alpha\sqrt{n-2}\sum_{j=1}^{n-1} |\mathrm{Re}\, z_j|, \ \textit{if } \Lambda(\alpha, z_1, ..., z_n) < 0.$$

Proof. It is easy to check that $1-2\alpha \geqslant 0$ for $n \geqslant 4$. Applying Lemmas 3 and 4 to $C(P')$ and the Hermitian matrices $A_1, B_1, B_2, \dots, B_{n-1}, A_3$, we obtain, in view of our analysis in the proof of Theorem 1, that

$$\begin{aligned}
&\sum_{j=1}^{k} \mathrm{Re}\, w_j = \sum_{j=1}^{k} \mathrm{Re}\lambda_j(C(P')) \leqslant \sum_{j=1}^{k} \lambda_j(\mathrm{Re}\, C(P'))\\
&\leqslant \sum_{j=1}^{k} \lambda_j(A_1) + \sum_{j=1}^{k} \lambda_j(B_1) + \sum_{j=1}^{k} \lambda_j(B_2) + \ldots + \sum_{j=1}^{k} \lambda_j(B_{n-1}) + \sum_{j=1}^{k} \lambda_j(A_3)\\
&= \begin{cases} (1-2\alpha)\sum_{j=1}^{k} \mathrm{Re}\, z_j + \alpha\sqrt{n-2}\sum_{j=1}^{n-1} |\mathrm{Re}\, z_j| + (n-1)\Lambda(\alpha, z_1, ..., z_n), \\ \qquad \text{if } \Lambda(\alpha, z_1, ..., z_n) \geqslant 0; \\ \\ (1-2\alpha)\sum_{j=1}^{k} \mathrm{Re}\, z_j + \alpha\sqrt{n-2}\sum_{j=1}^{n-1} |\mathrm{Re}\, z_j|, \text{ if } \Lambda(\alpha, z_1, ..., z_n) < 0 \end{cases}
\end{aligned}$$

for $k = 1, 2, \dots, n-2$. These inequalities, statement 1, and remark 1 complete the proof of the theorem. □

Theorems 1 and 2 supplement theorems 1 and 2 of the paper [1] respectively. And the new inequalities provided here improve the ones of [1] in the case where some roots of the polynomial P have imaginary parts with large moduli. In fact, the results of [1] involve also $|z_i|$, whereas the new bounds involve only the real parts of z_i.

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