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On Yield Criterion Choice in Thermoelastoplastic Problems

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Abstract

The present study is devoted to the boundary value problem of coupled thermoelastoplasticity. The temperature depended yield criterion and Duhamel Neumann constitutive equation was used. The material subjected to uneven heat treatment under plane strain frameworks was considered. The new analytical solution of the problem of uneven heat treatment of the thermoelasticplastic hollow cylinder was constructed within of the thermal stresses and plastic flow theories. The three different yield criteria were used. The original numerical scheme for calculations of temperature stresses and plastic strains in the frameworks of the von Mises yield criterion was developed and implemented. The characteristics of the plastic flow in the heating domain according to the yield criterion selection are eliminated. Constructed solutions of the boundary value problems were compared and analyzed for yield criteia of von Mises, Tresca and Ishlinsky-Ivlev.

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1. Introduction.

The thermal stresses and strains computing in the material under intensive irreversible deformation is one of the actual mathematical problem of modern solid mechanics. Calculation of temperature stress inside a cylindrical bodies is primarily needed to the stress-strain state investigations in the pipes, shafts, couplings and other cylindrical metal products subjected to intensive heat treatment. The precise residual stress computation taking into account the plastic

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flows can more accurately predict the final tightness of impacting bodies, for example, in shrink fitting process. High temperature gradient can cause a plastic flow in the metallic materials exposed to intensive thermomechanical loading, for example, in heat engines, turbines, nuclear reactors, and others. Emerging with residual stresses and strains changes the strength and the final geometric characteristics of the products.

The von Mises yield criterion is used in the majority of numerical scheme of the thermal stresses and strains calculations[1]. However, using such schemes for a strong non-stationary processes of irreversible deforming in a permanent rapid changing of temperature gradient can lead to an accuracy of numerical computations decreasing[2]. To verify the correctness and accuracy of the numerical scheme the exact analytical solutions due to piecewise linear yield criteria can be used[3-7].

Among of the earliest problems which have been solved in the frameworks of the perfect thermoalstoplasticity were ones concerning to the calculation of unsteady thermal stresses and strains arising in the elastic-plastic spherically symmetric bodies[8,9]. It has been shown that the process of unsteady heat conduction may lead to the appearance, disappearance and the growth of plastic flow domain. A number of problems within plane stress and Tresca yield criterion frameworks was solved by the theory of perfect plasticity[11,12] and the linear hardening one[13,14]. In depth studies of statement problem correctness for temperature dependent yield stress in thin plates and discs are given in[15-23]. The problem solution in frameworks of plane plastic strain hypothesis and for Tresca yield criterion are in detail considered in[24-31]. It should be noted that taking into account the yield criterion depending on the temperature, some boundary value problems under Tresca yield criterion don't have a solution[12,14,21,31].

This study presents a new analytical solution for the classical statement of the boundary value problem of a thick-walled tube deformation under the non-uniform thermal treatment. The classical solutions in the framework of the Tresca yield criterion[29] are generalized due to the yield strength depending on of the temperature. New analytical solutions of axisymmetric problems were obtained for case of the piecewise linear Ishlinsky-Ivlev yield criterion[3,4,32]. Numerical and graphical comparison of the stress-strain state parameters calculated under the von Mises, Tresca and Ishlinsky-Ivlev yield criteria was carried out.

2. Stress-strain state of a hollow cylinder under non-uniform thermal exposure.

Let consider hollow sustainable material cylinder with inner and outer radii R_1 and R_2 respectively. The strains arising in cylinder are infinitesimal and additively compound from reversible (elastic) e_{ij} and irreversible (plastic) p_{ij} parts

$$d_{rr} = e_{rr} + p_{rr} = u_{r,r}, \qquad d_{\varphi\varphi} = e_{\varphi\varphi} + p_{\varphi\varphi} = \frac{u_r}{r}, \qquad d_{zz} = e_{zz} + p_{zz} = 0. \tag{1}$$

Here u_r is the radial component of the displacement vector, the comma denotes the partial derivative with respect to the corresponding spatial coordinates.

The free thermal expansion conditions in the cylindrical coordinate system are given by:

$$\sigma_{rr}(R_1) = 0\,, \ \sigma_{rr}(R_2) = 0\,. \tag{2}$$

The Duhamel-Neumann constitutive equations[33] for isotropic materials are read:

$$\begin{aligned}
\sigma_{rr} &= (\lambda + 2\mu)e_{rr} + \lambda(e_{\varphi\varphi} + e_{zz}) - (3\lambda + 2\mu)\Delta,\\
\sigma_{\varphi\varphi} &= (\lambda + 2\mu)e_{\varphi\varphi} + \lambda(e_{rr} + e_{zz}) - (3\lambda + 2\mu)\Delta,\\
\sigma_{zz} &= (\lambda + 2\mu)e_{zz} + \lambda(e_{\varphi\varphi} + e_{rr}) - (3\lambda + 2\mu)\Delta.
\end{aligned} \tag{3}$$

Herein λ, μ denote the constitutive constants (Lame modulus), Δ is the linear thermal expansion strain.

The equilibrium equation and the continuity equation in the cylindrical symmetry case can transformed by

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \qquad d_{\varphi\varphi,r} + \frac{d_{\varphi\varphi} - d_{rr}}{r} = 0. \tag{4}$$

For thermoelastic equilibrium frameworks the equation $p_{ij} = 0$ is satisfied. Thus and the components of stress tensor and displacement vector can be derived by integrating of eq. (4) in form

$$\begin{aligned}
\sigma_{rr} &= -\frac{2\omega}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho + A(t) + \frac{B(t)}{r^2}, \\
\sigma_{\varphi\varphi} &= \frac{2\omega}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho - 2\omega\Delta(r,t) + A(t) - \frac{B(t)}{r^2}, \\
\sigma_{zz} &= -2\omega\Delta(r,t) + \frac{\lambda A(t)}{(\lambda+\mu)}, \\
u_r &= \frac{\omega}{\mu r}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho + \frac{A(t)r}{2(\lambda+\mu)} - \frac{B(t)}{2\mu r}.
\end{aligned} \tag{5}$$

Here $\omega = \dfrac{\mu(3\lambda+2\mu)}{(\lambda+2\mu)}$, $A(t)$, $B(t)$ are the time dependent function, obtained from the boundary conditions (2).
The linear thermal expansion can computed by

$$\Delta(r,t) = \alpha(T(r,t) - T_0) \tag{6}$$

where α denotes the linear thermal expansion coefficient, $T(r,t)$ is an actual temperature at the given point, T_0 is the referential temperature at initial free state. The temperature field can be obtained by the integrating of the heat conduction equation under given boundary conditions. We assume that the temperature of the outer cylindrical surface is given constant T_0, and the temperature of the inner cylindrical surface depend on time t :

$$T(r,t) - T_0 = \frac{t\ln(r/R_2)}{\ln(R_1/R_2)} \tag{7}$$

The temperature field at different time is shown on Fig.1.

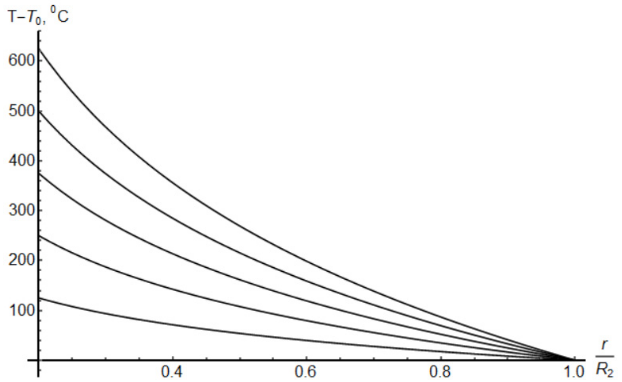

Figure 1. Temperature field at different time inside the considering cylinder $R_1/R_2 = 0.2$.

The gradual increase of the temperature gradient results in a change in the thermal stress value, whereby it becomes possible the irreversible deformation. The plastic flow process is coupled with the yield criteria satisfaction.

The most widely used in solid mechanics the following three yield criteria, two of which are piecewise linear ones Tresca yield criterion (maximum tangential stress criterion):

$$f = \max\left(\left|\sigma_1 - \sigma_2\right|, \left|\sigma_2 - \sigma_3\right|, \left|\sigma_3 - \sigma_1\right|\right) - 2k = 0, \tag{8}$$

Ishlinsky-Ivlev yield criterion (maximum reduced stress criterion):

$$f = \max\left(\left|\sigma_1 - \sigma\right|, \left|\sigma_2 - \sigma\right|, \left|\sigma_3 - \sigma\right|\right) - \frac{4}{3}k = 0, \tag{9}$$

von Mises yield criterion (maximum equivalent tensile stress criterion)

$$f = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 8k^2 = 0 \tag{10}$$

Material of cylinder reversibly deforms, if the inequality $f < 0$ is valid. In equations (8) – (10) we use following notation: σ_i are the principal stress values, $\sigma = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ is the hydrostatic stress, k is the yield stress in pure shear. We ca assume for yield stress following dependence[16] on thermal expansion:

$$k(\Delta) = k_0(1 - \beta\Delta), \tag{11}$$

wherein k_0 is the yield stress at the referential temperature T_0, β denotes the constitutive constant specifying the rate of yield stress according to temperature rate, which can be experimentally obtained.

The yield criterion is stated the plastic potential due to von Mises maximum principle. That implicit the associated flow rule[5-7] as the general constitutive equation of the flow theory

$$dp_{ij} = d\varsigma \frac{\partial f}{\partial \sigma_{ij}}. \tag{12a}$$

Herein dp_{ij} denote the plastic strains increments, ς is the undefined non-negative Lagrange multiplier.

Yield criteria (8) – (10) in the Haigh–Westergaard stress space can be interpreted as some surface at which solids manifest plastic properties. In particular, the Tresca and Ishlinsky-Ivlev yield criteria in in the Haigh–Westergaard stress space are presented as a hexagonal prism inclined to the coordinate axes, and the von Mises one is a cylinder. The projections of the Tresca and the Ishlinsky-Ivlev yield criteria on deviatoric plane (Fig. 2) are the regular hexagons with a center lying on the hydrostatic axis, and the similar projection of von Mises yield criteria is the circle of radius[32] $\dfrac{2\sqrt{2}k}{\sqrt{3}}$.

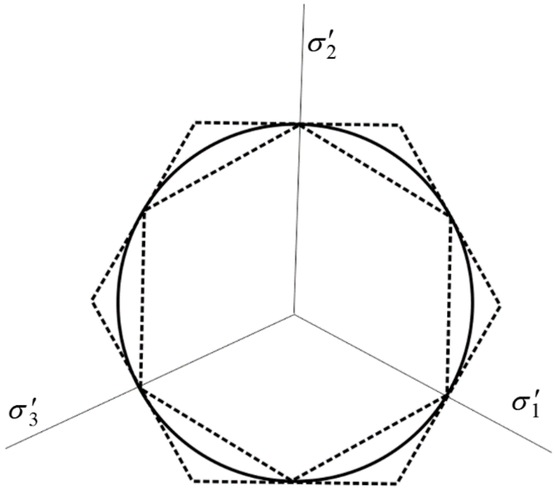

Figure 2. Yield criteria in deviatory plane. The circle is the von Mises yield criterion, inscribed hexagon is the Tresca yield criterion, escribed hexagon is the Ishlinsky-Ivlev yield criterion, σ_i' are the projections of the principal stresses at deviatory plane.

For considering problem in each facet of piecewise linear yield criterion corresponds to the equation that allows to analytically determine the stress-strain state parameters. Fig. 3 illustrates the various forms of Tresca a) and Ishlinsky-Ivlev b) yield criteria for various facet of hexagonal prisms.

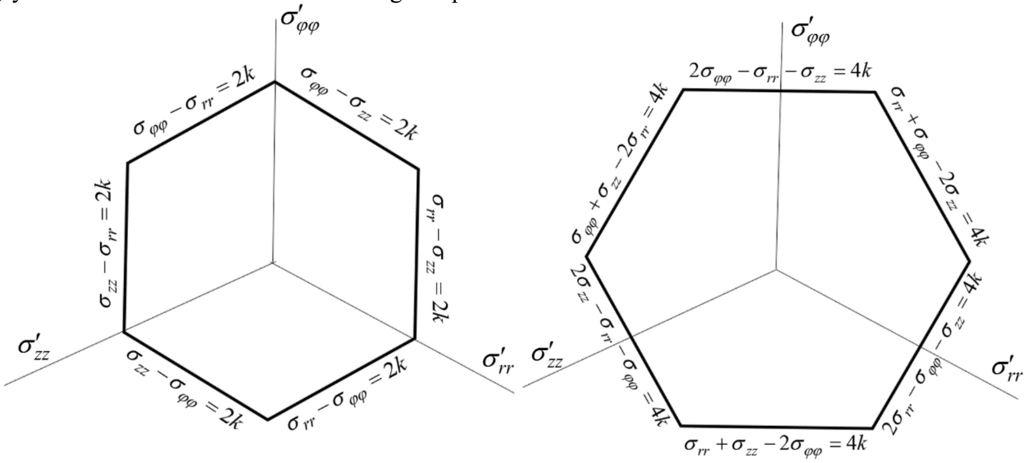

Figure. 3. Stresses equation at different edge of the piecewise yield surfaces: a) Tresca yield criterion; b) Ishlinshky-Ivlev yield criterion.

If the stress-strain state corresponds to the edge of considered yield criterion, when two equation are valid f_1, f_2, and we can obtain for associated flow rule following equation[6]

$$dp_{ij} = d\varsigma_1 \frac{\partial f_1}{\partial \sigma_{ij}} + d\varsigma_2 \frac{\partial f_2}{\partial \sigma_{ij}}. \tag{12}$$

2. Stress-strain state of the termoelastoplastic cylinder under Tresca yield criterion.

We consider boundary value problem of the deformation of cylinder within frameworks of the Tresca yield criterion. Plastic flow starts at time $t = t_1$ under condition shown on Fig. 3a. This criterion is satisfying at inner cylindrical surface by equation:

$$\sigma_{rr} - \sigma_{zz} = 2k \tag{13}$$

For times $t > t_1$ closely to the inner surface the plastic domain $R_1 < r < a_1$ is formed, $a_1(t)$ denotes the elastoplastic boundary which separates plastic flow domain from the thermoelastic one $a_1 < r < R_2$.

The following equations are derived by the plastic flow rule and eq. (13):

$$\begin{gathered} dp_{rr} = d\varsigma, \quad dp_{zz} = -d\varsigma, \quad dp_{\varphi\varphi} = 0. \\ p_{rr} + p_{zz} = 0, \quad p_{\varphi\varphi} = 0. \end{gathered} \tag{14}$$

We obtain differential equation for radial displacement from equations (3), (4), (13), and (14):

$$\begin{gathered} (ru_{r,r})_{,r} - \frac{\eta^2 u_r}{r} + \frac{(rk)_{,r}}{(\lambda+\mu)} - \frac{\gamma r \Delta_{,r}}{\mu} = 0, \\ \gamma = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}, \quad \eta = \sqrt{\frac{(\lambda+2\mu)}{(\lambda+\mu)}}. \end{gathered} \tag{15}$$

One can derive equation for displacement and plastic strain by integrating eq. (15):

$$\begin{aligned} u_r &= \frac{\gamma}{2\mu\eta}\left(\frac{(\eta+1)}{r^\eta}\int_{R_1}^{r}\rho^\eta\Delta(\rho,t)d\rho + (\eta-1)r^\eta\int_{R_1}^{r}\frac{\Delta(\rho,t)}{\rho^\eta}d\rho\right) - \\ &- \frac{1}{2(\lambda+\mu)}\left(\frac{1}{r^\eta}\int_{R_1}^{r}\rho^\eta k(\rho,t)d\rho + r^\eta\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^\eta}d\rho\right) + C(t)r^\eta + \frac{D(t)}{r^\eta}, \\ p_{rr} &= \frac{\eta}{4(\lambda+\mu)}\left(\frac{1}{r^{\eta+1}}\int_{R_1}^{r}\rho^\eta k(\rho,t)d\rho + r^{\eta-1}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^\eta}d\rho\right) + \frac{\gamma\Delta(r,t)}{2\mu} - \frac{\eta^2 k(r,t)}{2\mu} + \\ &+ \frac{\gamma}{4\mu}\left(r^{\eta-1}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^\eta}d\rho - \frac{(\eta+1)}{r^{\eta+1}}\int_{R_1}^{r}\rho^\eta\Delta(\rho,t)d\rho\right) + \frac{\eta r^{\eta-1}C(t)}{2} - \frac{\eta D(t)}{2r^{\eta+1}}, \end{aligned} \tag{16}$$

Calculate the thermal stresses in plastic domain $R_1 < r < a_1$ due to eq. (16):

$$\sigma_{rr} = \sigma_{zz} + 2k(r,t) = \frac{\gamma}{2\eta\mu}\left((\eta-1)\nu_1 r^{\eta-1}\int_{R_1}^{r} \frac{\Delta(\rho,t)}{\rho^{\eta}} d\rho - \frac{(\eta+1)\nu_2}{r^{\eta+1}}\int_{R_1}^{r} \rho^{\eta}\Delta(\rho,t)d\rho \right) -$$
$$- \frac{1}{2(\lambda+\mu)}\left(\nu_1 r^{\eta-1}\int_{R_1}^{r} \frac{k(\rho,t)}{\rho^{\eta}} d\rho - \frac{\nu_2}{r^{\eta+1}}\int_{R_1}^{r} \rho^{\eta} k(\rho,t)d\rho \right) + \nu_1 r^{\eta-1}C(t) - \frac{\nu_2 D(t)}{r^{\eta+1}},$$
$$\sigma_{\varphi\varphi} = \frac{\gamma}{2\eta\mu}\left((\eta-1)\upsilon_1 r^{\eta-1}\int_{R_1}^{r} \frac{\Delta(\rho,t)}{\rho^{\eta}} d\rho - \frac{(\eta+1)\upsilon_2}{r^{\eta+1}}\int_{R_1}^{r} \rho^{\eta}\Delta(\rho,t)d\rho \right) - \gamma\Delta(r,t) - \frac{\lambda k(r,t)}{(\lambda+\mu)} - \tag{17}$$
$$- \frac{1}{2(\lambda+\mu)}\left(\upsilon_1 r^{\eta-1}\int_{R_1}^{r} \frac{k(\rho,t)}{\rho^{\eta}} d\rho - \frac{\upsilon_2}{r^{\eta+1}}\int_{R_1}^{r} \rho^{\eta} k(\rho,t)d\rho \right) + \upsilon_1 r^{\eta-1}C(t) - \frac{\upsilon_2 D(t)}{r^{\eta+1}}.$$
$$\nu_1 = (\lambda+\eta\lambda+\eta\mu), \quad \nu_2 = (\eta\lambda-\lambda+\eta\mu), \quad \upsilon_1 = (\lambda+\eta\lambda+2\mu), \quad \upsilon_2 = (\lambda-\eta\lambda+2\mu).$$

The unknown functions presented in the eqs. (5), (16), (17) are found from the boundary conditions (2) and the continuity conditions of radial stresses and displacement at the elastic-plastic boundary. The position of the elastic-plastic boundary a_1 for a given time t is computed by the equation $p_{rr}(a_1,t)=0$.

At a certain time $t=t_2$ if the temperature still increase at inner cylinder surface the following conditions are satisfied:

$$\sigma_{rr} - \sigma_{zz} = 2k, \quad \sigma_{rr} - \sigma_{\varphi\varphi} = 2k. \tag{18}$$

This fact means the complete plasticity state. In time $t>t_2$ the new elasto-plastic boundary a_2 separates from the inner cylinder surface, which discriminates complete plasticity domain $R_1<r<a_2$ and plastic flow one $a_2<r<a_1$. Both plastic boundaries move to outer cylinder surface. We found stresses in complete plasticity domain by integrating equilibrium equation (5) taking into account eq. (18):

$$\sigma_{rr} = -2\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho + E(t), \quad \sigma_{\varphi\varphi} = \sigma_{zz} = -2\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - 2k(r,t) + E(t). \tag{19}$$

The equations for plastic strains follow from the associated plastic flow rule (12b) and eqs (18)

$$dp_{rr} = d\varsigma_1 + d\varsigma_2, \quad dp_{zz} = -d\varsigma_1, \quad dp_{\varphi\varphi} = -d\varsigma_2.$$
$$p_{rr} + p_{\varphi\varphi} + p_{zz} = 0. \tag{20}$$

Equation for radial displacement inside complete plasticity domain $R_1<r<a_2$ can be derived by eqs (1), (3), (19), (20)

$$u_r = -\frac{1}{(3\lambda+2\mu)}\left(\frac{1}{r}\int_{R_1}^{r} k(\rho,t)\rho d\rho + 3r\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - \frac{3rE(t)}{2} \right) + \frac{3}{r}\int_{R_1}^{r} \Delta(\rho,t)\rho d\rho + \frac{F(t)}{r}. \tag{21}$$

Stress-strain state parameters at other domains one can compute by the eqs (5), (16), (17) with new unknown functions A, B, C, D, and E, F which calculated by boundary conditions (2) and continuity conditions for stresses and displacement.

Let find the plastic strains in complete plasticity domain from eq (20)

$$\begin{aligned}
p_{rr} &= \frac{1}{(3\lambda+2\mu)}\left(\frac{1}{r^2}\int_{R_1}^{r} k(\rho,t)\rho d\rho - \int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho + \frac{E(t)}{2}\right) - \frac{2k(r,t)}{\gamma} - \frac{3}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho + 2\Delta(r,t) - \frac{F(t)}{r^2},\\
p_{\varphi\varphi} &= -\frac{1}{(3\lambda+2\mu)}\left(\frac{1}{r^2}\int_{R_1}^{r} k(\rho,t)\rho d\rho + \int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho - \frac{E(t)}{2}\right) + \frac{k(r,t)}{\gamma} + \frac{3}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho - \Delta(r,t) + \frac{F(t)}{r^2}, \quad (22)\\
p_{zz} &= \frac{1}{(3\lambda+2\mu)}\left(2\int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho - E(t)\right) + \frac{k(r,t)}{\gamma} - \Delta(r,t).
\end{aligned}$$

Elasto-plastic boundary positions are calculated by solutions of eqs $p_{rr}(a_1,t)=0$, $p_{\varphi\varphi}(a_2,t)=0$.

3. Stress-strain state of the termoelastoplastic cylinder under Ishlinsky-Ivlev yield criterion.

Let once more consider the boundary value problem of the deformation of cylinder, but the plastic potential chose in Ishlinsky-Ivlev's form (see Fig. 3b). Plastic flow starts again at time $t=t_1$ at inner cylinder surface:

$$2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{zz} = 2k \quad (23)$$

For arbitrary $t > t_1$ the plastic domain occupies one $R_1 < r < a_1$.

In this case the equations for plastic strains are read due to eqs (23) and (12a)

$$\begin{aligned}
&dp_{rr} = 2d\varsigma, \quad dp_{zz} = -d\varsigma, \quad dp_{\varphi\varphi} = -d\varsigma.\\
&p_{rr} + 2p_{zz} = 0, \quad p_{\varphi\varphi} = p_{zz}.
\end{aligned} \quad (24)$$

Differential equation for radial displacement one can obtain from (1), (3), (4), (23), (24).

$$(ru_{r,r})_{,r} - \chi^2\frac{u_r}{r} + \frac{2(3k+2rk_{,r})}{(3\lambda+2\mu)} - 3r\Delta_{,r} = 0, \quad \chi = \sqrt{\frac{(3\lambda+5\mu)}{(3\lambda+2\mu)}}. \quad (25)$$

Thus integrate eq (25) and calculate radial component of displacement vector in the domain $R_1 < r < a_1$:

$$\begin{aligned}
u_r &= -\frac{1}{\chi(3\lambda+2\mu)}\left(r^{\chi}(2\chi+1)\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^{\chi}}d\rho + \frac{(2\chi-1)}{r^{\chi}}\int_{R_1}^{r}\rho^{\chi}k(\rho,t)d\rho\right) +\\
&+ \frac{3}{2\chi}\left(r^{\chi}(\chi-1)\int_{R_1}^{r}\frac{\Delta(\rho,t)}{\rho^{\chi}}d\rho + \frac{(\chi+1)}{r^{\chi}}\int_{R_1}^{r}\rho^{\chi}\Delta(\rho,t)d\rho\right) + r^{\chi}C(t) + \frac{D(t)}{r^{\chi}}.
\end{aligned} \quad (26)$$

The equation for plastic strain using the eq. (26) is transformed as follows:

$$p_{rr} = \frac{(1-4\chi^2)}{3\chi(3\lambda+2\mu)}\left(r^{\chi-1}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^{\chi}}d\rho - \frac{1}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}k(\rho,t)d\rho \right) - \frac{2k(r,t)}{\omega} + 2\Delta(r,t) + \frac{1}{3}(2\chi-1)r^{\chi-1}C(t) +$$
$$+\frac{1}{2\chi}\left((2\chi-1)(\chi-1)r^{\chi-1}\int_{R_1}^{r}\frac{\Delta(\rho,t)}{\rho^{\chi}}d\rho - \frac{(2\chi+1)(\chi+1)}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}\Delta(\rho,t)d\rho \right) - \frac{(2\chi+1)D(t)}{3r^{\chi+1}}. \tag{27}$$

The thermal stresses therefore can be derived by (26), (27)

$$\sigma_{rr} = \frac{1}{3\chi}\left(r^{\chi-1}(\chi+1)(2\chi+1)\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^{\chi}}d\rho - \frac{(\chi-1)(2\chi-1)}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}k(\rho,t)d\rho \right) +$$
$$+(3\lambda+2\mu)\left(\frac{(\chi^2-1)}{2\chi}\left(r^{\chi-1}\int_{R_1}^{r}\frac{\Delta(\rho,t)}{\rho^{\chi}}d\rho - \frac{1}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}\Delta(\rho,t)d\rho \right) + \frac{1}{3}\left((\chi+1)r^{\chi-1}C(t) - \frac{(\chi-1)D(t)}{r^{\chi+1}} \right) \right),$$
$$\sigma_{\varphi\varphi} = \frac{1}{3\chi(3\lambda+2\mu)}\left(\frac{(2\chi-1)\xi_2}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}k(\rho,t)d\rho - r^{\chi-1}(2\chi+1)\xi_1\int_{R_1}^{r}\frac{k(\rho,t)}{\rho^{\chi}}d\rho \right) - 2k(r,t) + \tag{28}$$
$$+\frac{1}{2\chi}\left((\chi-1)\xi_1 r^{\chi-1}\int_{R_1}^{r}\frac{\Delta(\rho,t)}{\rho^{\chi}}d\rho - \frac{(\chi+1)\xi_2}{r^{\chi+1}}\int_{R_1}^{r}\rho^{\chi}\Delta(\rho,t)d\rho \right) + \frac{1}{3}\left(\xi_1 r^{\chi-1}C(t) - \frac{\xi_2 D(t)}{r^{\chi+1}} \right),$$
$$\sigma_{zz} = 2\sigma_{rr}(r,t) - \sigma_{\varphi\varphi}(r,t) - 4k(r,t),$$
$$\xi_1 = 3\lambda(\chi+1) + \mu(2\chi+5), \quad \xi_2 = 3\lambda(\chi-1) + \mu(2\chi-5).$$

Unknown time dependent functions A, B, C, D, in eqs (5), (26) – (28) one can find from boundary conditions (2) and stress-strain continuity conditions. The plastic boundary position we find from following equation $p_{rr}(a_1,t)=0$.

At time $t=t_2$ at elastic-plastic boundary $r=a_1$ the two equations are simultaneously valid (see Fig. 3 b):

$$2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{zz} = 4k, \quad \sigma_{rr} + \sigma_{\varphi\varphi} - 2\sigma_{zz} = 4k. \tag{29}$$

In this case the complete plasticity domain $a_1 < r < a_2$ for any time $t > t_2$ exist. Taking into account the eqs (5) and (18) derive the equations for thermal stresses in complete plasticity domain

$$\sigma_{rr} = E(t) - \frac{4}{3}\int_{a_1}^{r}\frac{k(\rho,t)}{\rho}d\rho, \; \sigma_{\varphi\varphi} = E(t) - \frac{4}{3}\left(\int_{a_1}^{r}\frac{k(\rho,t)}{\rho}d\rho + k(r,t) \right), \; \sigma_{zz} = E(t) - \frac{4}{3}\left(\int_{a_1}^{r}\frac{k(\rho,t)}{\rho}d\rho + 2k(r,t) \right). \tag{30}$$

Additional equations for plastic strains one ca obtain using by associated plastic flow rule (12b) and system (18)

$$dp_{rr} = 2d\varsigma_1 + d\varsigma_2, \quad dp_{zz} = -d\varsigma_1 - 2d\varsigma_2, \quad dp_{\varphi\varphi} = -d\varsigma_1 + d\varsigma_2.$$
$$p_{rr} + p_{\varphi\varphi} + p_{zz} = 0. \tag{31}$$

After that due to eqs (1), (3), (30), (31) the radial displacement function in domain $a_1 < r < a_2$ is rewrote as follows

$$u_r = \frac{3}{r}\int_{a_1}^{r}\Delta(\rho,t)\rho d\rho - \frac{2}{(3\lambda+2\mu)}\left(r\int_{a_1}^{r}\frac{k(\rho,t)}{\rho}d\rho + \frac{1}{r}\int_{a_1}^{r}k(\rho,t)\rho d\rho\right) + \frac{3E(t)r}{2(3\lambda+2\mu)} + \frac{F(t)}{r}. \quad (32)$$

Finally, the plastic strains in this domain are computed by

$$\begin{aligned}
p_{rr} &= \frac{1}{(3\lambda+2\mu)}\left(\frac{2}{r^2}\int_{R_1}^{r}k(\rho,t)\rho d\rho - \frac{2}{3}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho + \frac{E(t)}{2}\right) - \frac{2k(r,t)}{\omega} - \frac{3}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho + 2\Delta(r,t) - \frac{F(t)}{r^2},\\
p_{\varphi\varphi} &= -\frac{1}{(3\lambda+2\mu)}\left(\frac{2}{r^2}\int_{R_1}^{r}k(\rho,t)\rho d\rho + \frac{2}{3}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho - \frac{E(t)}{2} - \frac{4}{3}k(r,t)\right) + \frac{3}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho - \Delta(r,t) + \frac{F(t)}{r^2}, \quad (33)\\
p_{zz} &= \frac{2}{(3\lambda+2\mu)}\left(\frac{2}{3}\int_{R_1}^{r}\frac{k(\rho,t)}{\rho}d\rho - \frac{E(t)}{2} + \frac{2k(r,t)}{3\mu}\right) - \Delta(r,t).
\end{aligned}$$

From (33) it follows that the equation $p_{rr}(a_2,t)=0$ for determining of the elastic-plastic boundary position does not lead to the other plastic strains vanishing, which contradicts to the strains compatibility condition. It is obvious that in such a case the additional condition that is needed to specify the absence of the other plastic strains on the elastic-plastic boundary. However, in this case, the additional seventh equation overrides the system of six linear equations.

The resulting system of seven equations become an incompatible with respect to the variables A, B, C, D, E, F. This contradiction implies the impossibility of the existence of the complete plasticity domain near the reversible deformation domain. To resolve this problem, the existence of another plastic flow domain placed between the complete plasticity domain and elastic deformation domain has been suggested. The stress-strain state in this domain satisfies the following condition in the facet of the Ishlinsky-Ivlev yield surface (Fig. 3b)

$$\sigma_{rr} + \sigma_{\varphi\varphi} - 2\sigma_{zz} = 4k \quad (34)$$

At the time $t=t_2$ the boundaries a_2, a_3, are simultaneously arise and move with different rates to outer cylinder surface. Thus the boundary a_1 begins to move in the opposite direction. At any time $t>t_2$ the three plastic flow domains exist in cylinder: $R_1<r<a_1$ is the domain with stresses corresponding to Ishlinsky-Ivlev yield surface facet (23), $a_1<r<a_2$ is the domain corresponding to Ishlinsky-Ivlev yield surface edge (29), and $a_2<r<a_3$ is the domain corresponding to Ishlinsky-Ivlev yield surface facet (34).

Now, we compute the stress-strain state parameters in the plastic flow domain $a_2<r<a_3$. The equations can be obtained by associated plastic flow rule (12b), according to eq (34)

$$\begin{aligned}
&dp_{rr} = d\varsigma, \quad dp_{\varphi\varphi} = d\varsigma, \quad dp_{zz} = -2d\varsigma.\\
&2p_{rr} + p_{zz} = 0, \quad p_{rr} = p_{\varphi\varphi}.
\end{aligned} \quad (35)$$

Resulting differential equation for the radial displacement is derived by eqs (1), (3), (4), (34), (35)

$$(ru_{r,r})_{,r} - \frac{u_r}{r} + \frac{2rk_{,r}}{(3\lambda+5\mu)} - \frac{3r\Delta_{,r}}{\chi^2} = 0. \quad (36)$$

Integrating previous eq (36) one can compute

$$u_r = \frac{3}{\chi^2 r}\int_{a_2}^{r}\Delta(\rho,t)\rho d\rho - \frac{2}{(3\lambda+5\mu)r}\int_{a_2}^{r}k(\rho,t)\rho d\rho + G(t)r + \frac{H(t)}{r}. \quad (37)$$

The following equation for radial plastic strain is valid according to the just found displacement (37)

$$p_{rr} = \frac{\Delta(r,t)}{2\chi^2} - \frac{(\lambda+2\mu)k(r,t)}{\mu(3\lambda+5\mu)} + \frac{G(t)}{3}. \quad (38)$$

Stresses inside the plastic domain $a_2 < r < a_3$ are read

$$\begin{aligned}
\sigma_{rr} &= \frac{4\mu}{(3\lambda+5\mu)r^2}\int_{R_1}^{r}k(\rho,t)\rho d\rho - \frac{6\mu}{\chi^2 r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho + \frac{2}{3}(3\lambda+2\mu)G(t) - \frac{2\mu H(t)}{r^2},\\
\sigma_{\varphi\varphi} &= \frac{4\mu}{(3\lambda+5\mu)}\left(k(r,t) - \frac{1}{r^2}\int_{R_1}^{r}k(\rho,t)\rho d\rho\right) + \frac{6\mu}{\chi^2}\left(\frac{1}{r^2}\int_{R_1}^{r}\Delta(\rho,t)\rho d\rho - \Delta(r,t)\right) + \frac{2}{3}(3\lambda+2\mu)G(t) + \frac{2\mu H(t)}{r^2},\\
\sigma_{zz} &= \frac{2}{3}(3\lambda+2\mu)G(t) - \frac{3\mu\Delta(r,t)}{\chi^2} - \frac{2(3\lambda+4\mu)k(r,t)}{(3\lambda+5\mu)}.
\end{aligned} \quad (39)$$

As before we find the boundaries positions from eqs: $p_{zz}(a_1,t) = p_{\varphi\varphi}(a_1,t)$, $p_{rr}(a_2,t) = p_{\varphi\varphi}(a_1,t)$, $p_{rr}(a_3,t) = 0$.

3. Stress-strain state of the termoelastoplastic cylinder under von Mises yield criterion.

Let once more consider the above boundary value problem of the axisymmetric deformation, but the yield criterion chose in von Mises form (see Fig. 3b). Plastic flow starts at time $t = t_1$ at inner cylinder surface when the condition (10) is satisfied:

$$\sigma_{rr}^2 + \sigma_{\varphi\varphi}^2 + \sigma_{zz}^2 - \sigma_{rr}\sigma_{zz} - \sigma_{rr}\sigma_{\varphi\varphi} - \sigma_{zz}\sigma_{\varphi\varphi} - 4k^2 = 0. \quad (40)$$

The plastic strain increments are calculated by following equations taking into account plastic flow rule (12a) associated with yield surface (40)

$$\begin{aligned}
&dp_{rr} = 2d\zeta(2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{zz}), \quad dp_{\varphi\varphi} = 2d\zeta(2\sigma_{\varphi\varphi} - \sigma_{rr} - \sigma_{zz}), \quad dp_{zz} = 2d\zeta(2\sigma_{zz} - \sigma_{\varphi\varphi} - \sigma_{rr}).\\
&p_{rr} + p_{\varphi\varphi} + p_{zz} = 0.
\end{aligned} \quad (41)$$

The plastic strain increment dependence on stress tensor components makes it impossible to build the exact analytical solution of the problem in frameworks of the von Mises yield criterion. One-dimensional formulation of the problem allows us to find the stress-strain state parameters by means of numerical solution of differential equations with respect to $\zeta(r,t)$, $\sigma_{rr}(r,t)$.

Assume for plastic strain increments (40) $dp_{ij} = p_{ij} + p'_{ij}$, where p'_{ij} are the plastic strain computing at previous time. Then the reversible strains e_{rr}, $e_{\varphi\varphi}$ and stress σ_{zz} according to eqs (1), (3), (40) is derived by

$$e_{rr} = \frac{1}{4\mu(\Lambda + \mathrm{M})}\left(2\lambda\mu p'_{zz} + 2\Delta\mathrm{M} + (\Lambda + 2\mathrm{M})\sigma_{rr} - \Lambda\sigma_{\varphi\varphi}\right),$$
$$e_{\varphi\varphi} = \frac{1}{4\mu(\Lambda + \mathrm{M})}\left(2\lambda\mu p'_{zz} + 2\Delta\mathrm{M} + (\Lambda + 2\mathrm{M})\sigma_{\varphi\varphi} - \Lambda\sigma_{rr}\right), \tag{42}$$
$$\sigma_{zz} = \frac{1}{2(\Lambda + \mathrm{M})}\left((\Lambda + \mathrm{M} - \mu)(\sigma_{rr} + \sigma_{\varphi\varphi}) - 2\mu(3\lambda + 2\mu)(p'_{zz} + \Delta)\right),$$

wherein $\Lambda(r,t) = (\lambda + 12\lambda\mu d\varsigma(r,t))$, $\mathrm{M}(r,t) = (\mu + 8\mu^2 d\varsigma(r,t))$. If $d\varsigma = 0$ then the eqs (40) simulate the thermoelastic behavior ($\Lambda = \lambda$, $\mathrm{M} = \mu$).

We obtain the differential equation necessary for calculating the stress-strain state parameters by substituting eqs (41) and (42) in the the total strains equation (4)

$$p'_{\varphi\varphi} - p'_{rr} + rp'_{\varphi\varphi,r} + \frac{r}{2}(1 - \mu\Gamma)p'_{zz,r} + \frac{r}{4}(3\lambda + 2\mu)\left(p'_{zz} + \Delta + \frac{\sigma_{rr}}{(3\lambda + 2\mu)}\right)\mathrm{M}_{,r} + \frac{r}{2\lambda}\Lambda\Gamma(3\lambda + 2\mu)\Delta_{,r} +$$
$$+\left(\frac{3r\Lambda(1 + \mu\Gamma)}{4\lambda\mu} + \frac{r^2\mathrm{M}_{,r}}{8}\left(\Gamma^2 + \frac{3}{\mu^2}\right)\right)\sigma_{rr,r} + \frac{r^2\Lambda(1 + \mu\Gamma)}{4\lambda\mu}\sigma_{rr,rr} = 0, \tag{43}$$

where $\Gamma(r,t) = \dfrac{1}{\Lambda(r,t) + \mathrm{M}(r,t)}$.

Note that the equation (43) have the general solution (5) inside thermoelastic domain ($d\varsigma = 0$) without initial plastic strains $p'_{ij} = 0$. During the plastic flow, the equation (43) must be supplemented by the yield criterion (40) and the boundary conditions (2). The numerical solution of this system of equations at each time step allows us to calculate the values $d\varsigma$, σ_{rr} and to compute the stresses and irreversible strains. In these numerical calculations we used the successive approximations method. For this reason, the system of differential equations is reduced to a system of algebraic equations by standard finite-difference approximations of derivatives. The accuracy criterion of the solution is the condition validation $d\varsigma > 0$ [2] at each time step in each spatial node where the yield criterion is true (40).

4. The Comparison of Results.

Next, we discuss the comparison of thermal stresses in the hollow thermoelastoplastic cylinder calculated for different yield criteria at the same temperature field.

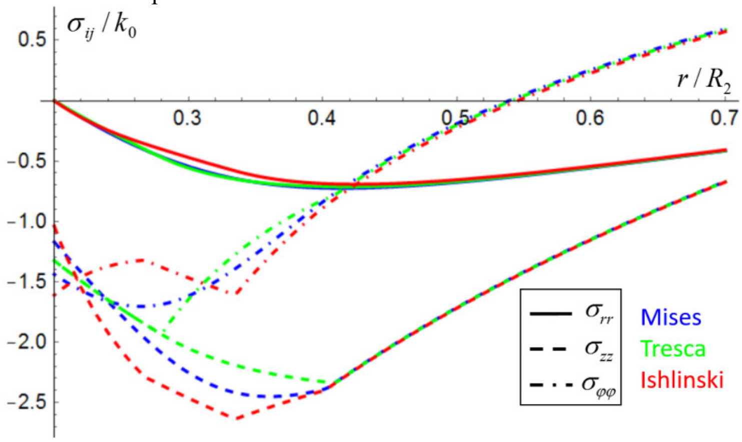

Figure. 4. The thermal stresses for various yield criteria.

Fig. 4 shows that the radial stress fields do not sufficient differ for various flow criteria. This means that, for example, in shrink fitting numerical simulation[13,16,17,22,26], when the major stress part is the contact pressure, it is possible to use any one of presented here yield criterion. Note that for various temperature gradient rate and product geometry, the only plastic flow domain exists under Ishlinsky-Ivlev yield criterion, but under Tresca yield criterion usually two plastic domains are presented. Therefore, the problem solution for the piecewise-linear yield criteria and given boundary conditions have a simpler form.

It should also be noted that the stresses values $\sigma_{\varphi\varphi}$, σ_{zz} (Fig. 4) for von Mises yield criterion are placed between stresses for piecewise yield criteria. This feature is absolutely consistent with the spatial position of von Mises yield surface (see Fig. 2). Therefore, we can construct the analytical solution which is the median one of the Tresca and Ishlinsky-Ivlev yield criteria solutions. Fig. 11 shows the comparison of the resulting arithmetical combination of two analytical solutions and numerical solution for the von Mises yield criterion one.

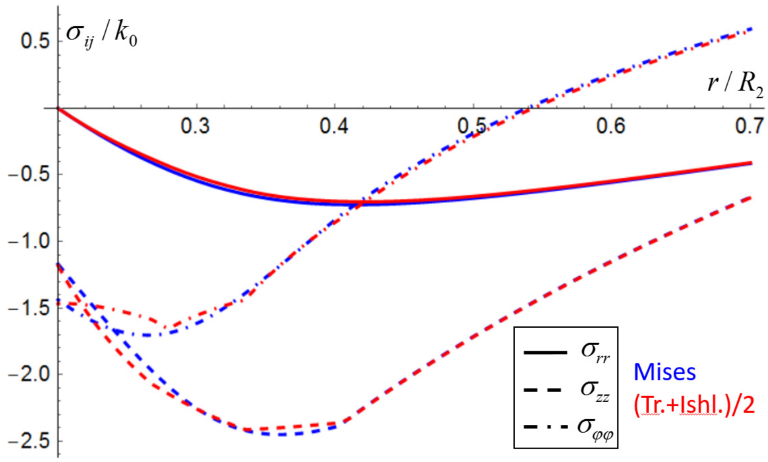

Figure. 5. The analytical median stress corresponding to Tresca and Ishlinsky-Ivlev yield criterion and numerical results corresponding von Mises one.

Conclusion

On the basis of simulation results, we can draw the following conclusion: the differences in the results obtained for piecewise-linear yield criteria (Tresca, Ishlinsky-Ivlev) and von Mises one can be minimal if we consider the linear combinations of the solutions[32]: in particular, the average value between the stresses satisfying the Tresca and Ishllinsky-Ivlev yield criteria in plane strain frameworks is very similar the stresses satisfying the von Mises yield criterion. This fact should be taken into account when the use of piecewise linear plasticity conditions provides a simpler solution. Particularly the approach is relevant for problems of the thermal conductivity processes when the elastic-plastic and unloading boundaries localization is difficult because of significant unevenness and unsteadiness thermal stresses and strains fields.

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