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Calculation of the Residual Stresses of Hollow Cylinder under Unsteady Thermal Action

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***Abstract*— The present study is devoted to the problem of unsteady thermal action on hollow elastoplastic cylinder. This physical process is mathematically proposed as a quasi-static process of the uniform thermal expansion of the hollow elastoplastic cylinder. The generalized Prandtl-Reuss thermoelastoplastic model is used. The effect of the non-stationary temperature gradient on the residual stresses field formation was investigated under the condition that the yield stress depends on a temperature. The borders of the irreversible deformation domain and unloading domain are computed. The level of residual stresses and strains are calculated after the final cooling of the cylinder.**

***Index Terms*— elasticity, heat conduction, plasticity, residual strain, residual stress**

I. Introduction

THERMAL stresses have a significant effect on details of the various mechanisms operating in the high temperature gradients. Residual strains and stresses form in a non-stationary temperature field variation. Accurate determination of the geometry and strength characteristics of the concerned materials it is needed because of such strains and stresses. As it's well known, temperature influences on the material yield stress by increasing probability of the appearance of the plastic deformations. Detailed analysis of stress-strain state (SSS) of cylindrical bodies under the steady-state thermal action and external pressure was considered [1]. The features of formation of non-reversible deformation fields were observed for the finite size solid cylinder having inner heat source [2]. The numerical solutions of the shrink fit problem for discs were compared using the Mises yield condition and Tresca yield condition [3]. The analytical solution of the shrink fit problem for discs considered in condition of yield stress dependence on temperature [4]. In [5] the problem of formation of residual stresses in a thin plate made of an elastoplastic material under a given thermal action was solved.

Here we present the exact solution of the problem of formation of residual stresses under the assumption that the connection between the processes of heat conduction and deformation under the conditions of intensive thermal action can be neglected. Thus, the calculations can be performed in the framework of the thermal stresses theory (see, for example, [6]) with the yield stress dependence on the temperature taken into account.

The features of residual strains and stresses formation for the load-free hollow cylinder with rapidly changing temperature gradient on the inner surface were investigated in this work. The possibility of the repeated plastic flow appearance in case of temperature field aligning was observed. The method for determining of the non-reversible deformations on the boundary between the plastic flow domain and the unloading domain was shown and the residual strains and stresses were calculated.

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II. Governing Equations

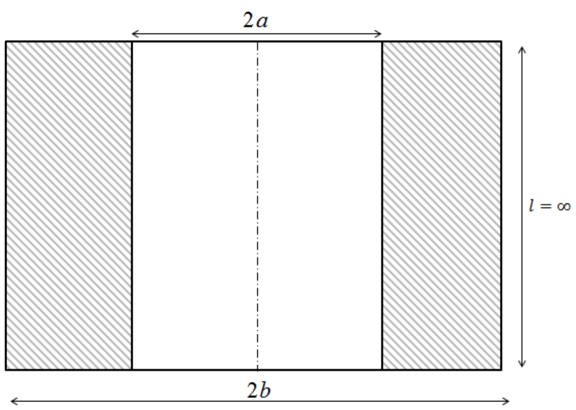

Fig. 1. Schematic of hollow solid cylinder.

A framework of thermoelastoplasticity (see [6] for details) is used throughout the paper. The schematic of infinitely long hollow cylinder with lateral surfaces devoid of loads is shown in Fig. 1. a, b are the inner and outer radiuses of hollow cylinder.

$$\begin{aligned} \sigma_{rr}(a,t) &= 0, \\ \sigma_{rr}(b,t) &= 0. \end{aligned} \tag{1}$$

At the initial time $t = 0$ the temperature of cylinder is $T(r,0) = T_0$. At the time $t > 0$ on the outer and inner surfaces of the cylinder the following conditions are satisfied:

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$$T(a,t) = T_k\left(1-\exp(-xt)\right),$$
$$T_{,r}(r,t)\big|_{r=b} = 0, \tag{2}$$

where x is the parameter defining the temperature rate increasing on the inner surface of the cylinder. The index after comma denotes differentiation with respect to the corresponding spatial coordinate. At the time $t \to \infty$ the determination of a constant temperature $T(a) = T_k$ follows from the equation (2).

Temperature field is described by the heat equation:

$$T_{,t} = \frac{\chi}{r}\left(rT_{,r}\right)_{,r}, \tag{3}$$

where χ is the thermal diffusivity.

A differentiation in concern the variable r is denoted by a comma herein and below.

Taking into consideration that infinitesimal strains arising due to the thermal action (3) relations for the radial and angular components of strain have the following form:

$$d_{rr} = u_{r,r} = e_{rr} + p_{rr},$$
$$d_{\varphi\varphi} = \frac{u_r}{r} = e_{\varphi\varphi} + p_{\varphi\varphi},$$

(4)

where u_r is the radial component of the displacement vector, e_i, p_i are the elastic and plastic components of strain tensor. Stresses are determined by the thermoelastic strains according to the Duhamel-Neumann Law [6]:

$$\sigma_{rr} = \omega e_{rr} + \lambda e_{\varphi\varphi} - \Delta,$$
$$\sigma_{\varphi\varphi} = \omega e_{\varphi\varphi} + \lambda e_{rr} - \Delta,$$
$$\sigma_{zz} = \lambda\left(e_{rr} + e_{\varphi\varphi}\right) - \Delta, \tag{5}$$
$$\Delta = \alpha(3\lambda + 2\mu)(T - T_0),$$
$$\omega = (\lambda + 2\mu),$$

where λ, μ are Lame parameters, α is the coefficient of linear thermal expansion.

Within the framework the considered problem the radial and angular stresses must satisfy the equilibrium equation and the corresponding values of the strains must satisfy the continuity condition:

$$\sigma_{\varphi\varphi} = \left(r\sigma_{rr}\right)_{,r},$$
$$d_{rr} = \left(rd_{\varphi\varphi}\right)_{,r}. \tag{6}$$

The Tresca condition is selected as the yield criteria [7]:

$$\max\{|\sigma_{rr} - \sigma_{\varphi\varphi}|, |\sigma_{rr} - \sigma_{zz}|, |\sigma_{\varphi\varphi} - \sigma_{zz}|\} = 2k(T), \tag{7}$$

where $k(T)$ is the yield stress at the corresponding temperature. For further calculations, we assume the simple linear dependence $k(T) = k_0(T_p - T)(T_p - T_0)$, where k_0 is the yield stress at the ambient temperature and T_p is the melting point.

III. SOLUTION

Both the analytical solution and the computational algorithms are existing for the heat equation (3) with boundary conditions (2). We assume that the temperature distribution is known.

The solution of the equilibrium equation for the problem of thermoelasticity [6] has the form:

$$u_r(r,t) = \frac{r}{\omega}F_a(r,t) + c_1(t)r + \frac{c_2(t)}{r},$$
$$\sigma_{rr}(r,t) = -\frac{2\mu}{\omega}F_a(r,t) + 2\gamma c_1(t) - \frac{2\mu}{r^2}c_2(t),$$
$$\sigma_{\varphi\varphi} = \frac{2\mu}{\omega}F_a(r,t) + 2\gamma c_1(t) + \frac{2\mu}{r^2}c_2(t) - 2\frac{\mu}{\omega}\Delta(r,t), \tag{8}$$
$$\sigma_{zz}(r,t) = -\frac{2\mu}{\omega}\Delta(r,t) + 2\lambda c_1(t),$$

where $\gamma = (\lambda + \mu)$, $F_y(r,t) = \frac{1}{r^2}\int_y^r \Delta(\rho,t)\rho d\rho$.

Functions $c_1(t)$, $c_2(t)$ are determined from the boundary conditions (1).

Increasing the temperature gradient leads to the satisfaction of the yield criteria (7) on the inner surface of the cylinder at the time t_p:

$$\sigma_{rr} - \sigma_{\varphi\varphi} = 2k. \tag{9}$$

At the time $t > t_p$ there is the plastic flow region $a < r < a_1(t)$, where $a_1(t)$ is the elastoplastic border. According to the associated flow rule the incompressibility condition follows from (9):

$$p_{rr} + p_{\varphi\varphi} = 0.$$

(10)

Using the assumption (4) and the Duhamel-Neumann Law (5) we obtain the differential equation for displacements for the case of plastic strains existence:

$$\omega\left(u_{r,r} + \frac{u_r}{r}\right)_{,r} = 2\mu\left(2\frac{p_{rr}}{r} + p_{rr,r}\right) + \Delta_{,r}.$$

(11)

The dependence between displacements and plastic strains also follows from the condition (9):

$$p_{rr} = -p_{\varphi\varphi} = \frac{1}{2}\left(r\left(\frac{u_r}{r}\right)_{,r} - \mu^{-1}k\right).$$

(12)

Integrating the system of equations (11), (12) we found:

$$u_r(r,t) = \frac{r}{\gamma}\left(F_a(r,t) - G_a(r,t)\right) + c_3(t)r + \frac{c_4(t)}{r},$$
$$p_{rr}(r,t) = \frac{\Delta(r,t)}{2\gamma} - \frac{k(r,t)}{2\omega} - \frac{c_4(t)}{r^2} - \frac{F_a(r,t)}{\gamma},$$
$$G_y(r,t) = \int_y^r \rho^{-1}k(\rho,t)d\rho.$$

(13)

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The stresses in the plastic region can be found by integrating the system (6), (9):

$$\sigma_{rr}(r,t) = -G_a(r,t) + 2\gamma c_3(t),$$
$$\sigma_{\varphi\varphi}(r,t) = -G_a(r,t) + 2\gamma c_3(t) - 2k(r,t), \quad (14)$$
$$\sigma_{zz}(r,t) = -\frac{1}{\gamma}\left(G_a(r,t) + k(r,t)\right) + 2\lambda c_3(t) - \frac{\mu}{\gamma}\Delta.$$

In the elastic deformation region $a_1(t) \le r < b$ SSS is determined by the relations (8) previously obtained with accuracy to the new integration constants which together with the constants in the plastic region require its definition. For this it is needed to solve the system of linear equations in the form of boundary conditions (1) and continuity conditions of the radial stresses and displacements on the elastoplastic border $a_1(t)$ which is determined by the condition $p_{rr}(a_1, t) = 0$.

During the temperature field alignment, the non-reversible strains in the neighborhood of elastoplastic border continue to increase whereas on the inner surface its rate becomes equal to zero:

$$p_{rr,t}(r, t_r)\Big|_{r=a} = 0. \quad (15)$$

Relation (15) corresponds to the beginning of materials unloading, i.e. deformation process in which the yield condition (9) ceases to be satisfied. At the time $t > t_r$ the unloading region $a < r \le \tilde{a}_1(t)$ exists. Displacements and stresses are obtained by solving of the equilibrium equation (11):

$$u_r(r,t) = \frac{r}{\omega}F_a(r,t) + \frac{2\mu r}{\omega}H_a(r) + c_5(t)r + \frac{c_6(t)}{r},$$
$$\sigma_{rr}(r,t) = -\frac{2\mu}{\omega}F_a(r,t) + \frac{4\mu\gamma}{\omega}H_a(r) + 2\gamma c_5(t) - \\ -\frac{2\mu}{r^2}c_6(t),$$
$$\sigma_{\varphi\varphi}(r,t) = 2\mu F_a(r,t) - \frac{2\mu}{\omega}\Delta(r,t) + \frac{4\mu\gamma}{\omega}H_a(r) + \\ + 2\gamma c_5(t) + \frac{2\mu}{r^2}c_6(t) + \frac{4\mu\gamma}{\omega}\hat{p}_{rr}(r),$$
$$\sigma_{zz}(r,t) = \frac{2\lambda\mu}{\omega}\left(2H_a(r) + \hat{p}_{rr}(r)\right) - \frac{2\mu}{\omega}\Delta(r,t) + \\ + 2\lambda c_5(t),$$
$$H_y(r) = \int_y^r f(\rho)d\rho, \quad f(r) = r^{-1}\hat{p}_{rr}(r).$$
(16)

where $\hat{p}_{rr}(r)$ is the plastic strain captured at the given time on the unloading border which is defined by relation (15).

SSS in the region $\tilde{a}_1(t) < r \le a_1(t)$ and $a_1(t) < r \le b$ is determined by the previously obtained relations with an accuracy to the new integration constants. These constants with the constants from equations (16) are found from the system of linear equations describing the continuity of stresses and displacements on the regions boundaries. To obtain a function $\hat{p}_{rr}(r)$ and relations describing the displacement of the plastic flow border and the unloading border the following system of equations should be numerically solved:

$$\begin{cases} p_{rr,t}(\tilde{a}_1^i, t_i) = 0, \\ p_{rr}(\tilde{a}_1^i, t_i) = \tilde{a}_1^i f(\tilde{a}_1^i), \\ p_{rr}(a_1^i, t_i) = 0. \end{cases}$$
(17)

The first equation of the system defines the unloading border $\tilde{a}_1$, the second one defines the accumulated plastic strains, and the third one describes the elastoplastic border a_1. Subscript i indicates a time step.

TABLE
MATERIAL CONSTANTS

Symbol	Quantity	Value
a	internal radius of cylinder	0.1 m
b	external radius of cylinder	0.2 m
λ	Lamé constant	91.2×10^9 Pa
μ	Lamé constant (shear modulus)	42.9×10^9 Pa
k_0	yield stress at the ambient temperature	80×10^6 Pa
χ	thermal diffusivity	11.4×10^{-6} m^2/s
α	coefficient of linear thermal expansion	17×10^{-6} m^2/s
T_0	initial temperature	293 K
T_p	melting point	1357.77 K
T_k	final temperature	723 K

The integrand $f(r)$ of the integral H_y (16) can be replaced by the piecewise linear approximation at the each time step. It allows to submit this integral by the trapezoidal rule in the form:

$$H_{\tilde{a}_1^{i-1}}(\tilde{a}_1^i) = \frac{1}{2}\left(\tilde{a}_1^i - \tilde{a}_1^{i-1}\right)\left(f(\tilde{a}_1^i) - f(\tilde{a}_1^{i-1})\right) + \\ + H_{\tilde{a}_1^{i-2}}(\tilde{a}_1^{i-1}).$$
(18)

To solve the system (17) at the various time points using the approximation method we obtain the corresponding values of the deformation regions sizes and construct the function of the residual strain. The initial parameters of the system (17) have a form:

$\tilde{a}_1^0 = a_1^0 = a_1(t_r)$, $f(\tilde{a}_1^0) = a_1^0 d_{rr}(a_1(t_r), t_r)$.

IV. RESULTS AND DISCUSSION

The parameters corresponding to copper were used for calculations [9] are shown in Table:

The solution was found for different values of parameter x (heating rate) in (2). The plastic flow does not appear at small values of x and also at small temperature gradient. The appearance of the plastic flow near the border with inner surface of the cylinder is observed with increasing of

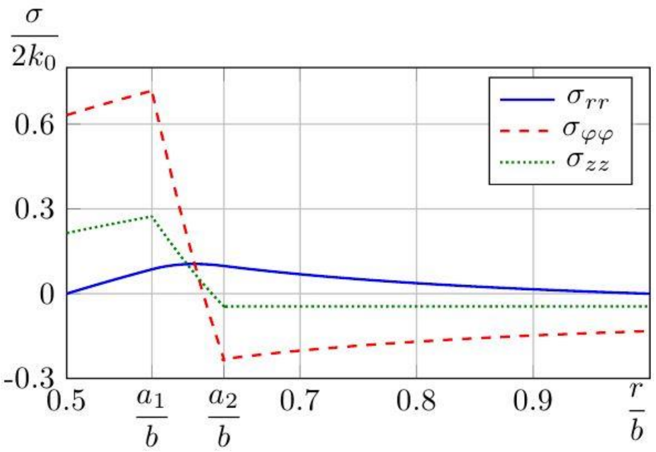

Fig. 2. The non-dimensional residual stresses distribution. The plastic flow border $a_1 = 0.57304$ and the repeated plastic flow border $a_2 = 0.6348$ are marked by vertical lines.

parameter x. Temperature stresses decrease and unloading

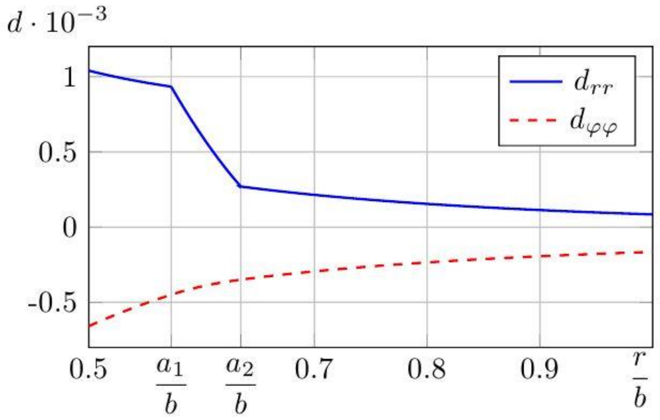

Fig. 3. The residual strains distribution. The plastic flow border $a_1 = 0.57304$ and the repeated plastic flow border $a_2 = 0.6348$ are marked by vertical lines.

region appears during the temperature gradient alignment. The unloading border eventually overtakes the elastoplastic border. The condition (9) with the opposite sign in front of the yield stress is satisfied on the inner surface of the cylinder in case of high values after the complete unloading of material. This implies the formation of plastic flow when the plastic deformation increases in the opposite direction, then so, that decreasing the value of residual strain accumulated during heating. The occurrence of repeated plastic flow is caused by high residual stresses and also by significant decrease of the yield stress. Fig. 2 illustrates the residual stresses under the repeated plastic flow, which were calculated in fully cooled cylinder.

Note that the stresses level is independent of the current temperature [4] and determined by the temperature gradient level. Consequently, the stresses distribution at complete heating of the cylinder to a maximum temperature coincides with the stress field at complete cooling. Fig. 3 illustrates the residual strains in case of the repeated plastic flow calculated at complete cooling of the cylinder.

V. Conclusion

The problem of unsteady thermal action on hollow elastoplastic cylinder has been considered. This physical process has been mathematically proposed as a quasi-static process of the uniform thermal expansion of the hollow elastoplastic cylinder. The generalized Prandtl-Reuss thermoelastoplastic model has been used. The effect of non-stationary temperature gradient on the residual stresses field formation on condition of dependence of the yield stress on temperature has been investigated. The resulting system has been analytically integrated. The border of irreversible deformation domain and unloading domain have been computed. The level of residual stresses and strains have been calculated after the final cooling of the cylinder. The results have been graphically presented.

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