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Thermal Stresses Computation under Toroidal Symmetry Conditions

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Abstract. The present study deals with the elastic plastic boundary value problems statements in toroidal coordinates. The basic model relations of the temperature stresses theory are furnished in a toroidal coordinate system. The computation problem of the stress-strain state of a hollow elastic-plastic torus subjected to uneven radial heating is considered. Analytical solutions are obtained for the stress and displacement fields in the thermoelastic deformation and plastic flow domains within the framework of the generalized plane stress state hypothesis. The possibility of applying the solutions obtained for the stress-strain state of a torus of arbitrary sizes under conditions of an axisymmetric thermal action is discussed.

INTRODUCTION

The thermal stresses theory studies the effect of a given temperature field on the stress-strain state evolution in the material. The particular interest of the researchers is the study of the processes of irreversible deformation being result of the influence of high temperature gradients. A number of boundary value problems have been solved for solid elastic-plastic material under conditions of central and axial symmetry of the thermal action. In particular, one-dimensional solutions are obtained for an thermoelastic-plastic ball see in depth studies for example in [1, 2, 3, 4, 5, 6]. The non-uniform temperature heating and the dependence of the yield stress on temperature were taking into account. The solutions in a cylindrical coordinate system in frameworks of axial symmetry conditions are obtained for various yield criteria describing the irreversible deformation of elastic-plastic thin disks [7, 8, 9, 10, 11], thick-wall pipes [12, 13, 14, 15, 16, 17, 18, 19, 20, 21], assembled circular structures manufacturing by the shrink fit method [9, 22, 23, 23]. Such thermomechanical problems and solutions are needed for simulation of the additive manufacturing technology processes and further exploitation of materials producing by such approach [5, 6, 24, 25, 26, 27, 28, 29, 30, 31, 32].

The plane strain or plane stress state hypotheses, as well as an axisymmetric distribution of the temperature field are assumed for above symmetrical problems statements allowing us to obtain reduced one-dimensional solutions describing the wide range of the thermoelastic behaviour of a material and its deformation during plastic flow and following unloading processes.

One of the poorly studied problem in the thermal stresses theory is the one of the solid deformation state possessing toroidal symmetry. The toroidal structures is widely used in such fields as magnetostatics, magnetohydrodynamics, controlled thermonuclear fusion, thermonuclear synthesis, etc. The temperature field influence can be significant in evaluating of the strength characteristics and deformations for such structures. The present paper is devoted to the problem of the stress-strain state computation of a hollow elastic-plastic torus subjected to a radial distribution of the temperature gradient.

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GOVERNING EQUATIONS IN TOROIDAL COORDINATES

Throughout the paper we will use conventional thermoelastic-plastic model akin to Prandtl and Reuss [33]. Infinitesimal deformations $d_{ij} = e_{ij} + p_{ij}$ with elastic e_{ij} and plastic p_{ij} compounds are derived due to the displacements u_i by following equations

$$
\begin{gathered}
d_{rr} = u_{r,r}, \quad d_{\theta\theta} = \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r}, \quad d_{\varphi\varphi} = \frac{u_r \sin(\theta) + u_\theta \cos(\theta)}{(R_0 + r\sin(\theta))} + \frac{u_{\varphi,\varphi}}{(R_0 + r\sin(\theta))}, \quad d_{r\theta} = \frac{1}{2}\left(\frac{u_{r,\theta}}{r} + u_{\theta,r} - \frac{u_\theta}{r}\right), \\
d_{r\varphi} = \frac{1}{2}\left(\frac{u_{r,\varphi}}{(R_0 + r\sin(\theta))} + u_{\varphi,r} - \frac{u_\varphi \sin(\theta)}{(R_0 + r\sin(\theta))}\right), \quad u_{\theta\varphi} = \frac{1}{2}\left(\frac{u_{\theta,\varphi}}{(R_0 + r\sin(\theta))} + u_{\varphi,\theta} - \frac{u_\varphi \cos(\theta)}{(R_0 + r\sin(\theta))}\right).
\end{gathered}
\tag{1}
$$

where indices after comma denote the partial derivative with respect to spatial coordinate.

The equilibrium equations can be formulated in terms of the stress tensor σ_{ij} by

$$
\begin{gathered}
\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{r\varphi,\varphi}}{(R_0 + r\sin(\theta))} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\sin(\theta)}{(R_0 + r\sin(\theta))}(\sigma_{rr} - \sigma_{\varphi\varphi} + ctg(\sigma_{r\theta})) = 0, \\
\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{\sigma_{\theta\varphi,\varphi}}{(R_0 + r\sin(\theta))} + \frac{2\sigma_{r\theta}}{r} + \frac{\sin(\theta)}{(R_0 + r\sin(\theta))}(\sigma_{r\theta} + ctg(\theta)(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})) = 0, \\
\sigma_{r\varphi,r} + \frac{\sigma_{\theta\varphi,\theta}}{r} + \frac{\sigma_{\varphi\varphi,\varphi}}{(R_0 + r\sin(\theta))} + \frac{\sigma_{r\varphi}}{r} + \frac{2\sin(\theta)}{(R_0 + r\sin(\theta))}(\sigma_{r\varphi} + ctg(\theta)\sigma_{\theta\varphi}) = 0
\end{gathered}
\tag{2}
$$

Constitutive stress-strain equations are read according to the Duhamel-Neumann thermoelastic constitutive law [33]

$$
\sigma_{ij} = \lambda\delta_{ij}(e_{rr} + e_{\theta\theta} + e_{\varphi\varphi}) - \alpha\delta_{ij}(3\lambda + 2\mu)(T - T_0) + 2\mu e_{ij}, \tag{3}
$$

where δ_{ij} is Kronecker delta; λ, μ are the Lame modulus; α denotes linear thermal expansion coefficient; $(T - T_0)$ defines difference between the actual and referential temperature of the material. Heat conduction equation in toroidal coordinates is furnished by

$$
T_{,rr} + \frac{(R_0 + 2r\sin(\theta))T_{,r}}{r(R_0 + r\sin(\theta))} + \frac{T_{,\theta\theta}}{r^2} + \frac{\cos(\theta)T_{,\theta}}{r(R_0 + r\sin(\theta))} + \frac{T_{,\varphi\varphi}}{(R_0 + r\sin(\theta))^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}. \tag{4}
$$

PROBLEM STATEMENT

Consider a hollow torus with the major radius R_0 and $r_1 < r < r_2$ (inner and outer radii). We also assume that the material is affected by the axisymmetric temperature distribution with respect to the Cartesian axis Z. Free thermal expansion condition is given on the inner and outer toroidal surfaces

$$
\sigma_{rr}(r_1, \theta) = 0, \quad \sigma_{r\theta}(r_1, \theta) = 0, \quad \sigma_{rr}(r_2, \theta) = 0, \quad \sigma_{r\theta}(r_2, \theta) = 0. \tag{5}
$$

At first the problem is considered under conditions of thermoelastic equilibrium ($d_{ij} = e_{ij}$). Numerical analysis of the boundary value problems shows us that the temperature field distribution essentially depends on the geometry of the torus. Note that the temperature field for small values of the parameter $\epsilon = r_2/R_0$ can be described by function depending only on radial coordinate. Moreover the toroidal symmetry becomes cylindrical when the parameter $\epsilon = r_2/R_0$ is tended to zero, which allows us to obtain one-dimensional analytical solutions. It is important to determine the admissible non-zero values of the parameter ϵ, for which the cylindrical solutions will satisfactorily describe two-dimensional numerical toroidal solutions for a given geometry.

The stationary heat conduction equation at $\epsilon = 0$ can be derived in form

$$
T_{,r} + rT_{,rr} = 0.
$$

with the boundary conditions $T(r_1, \theta) = T_k, \quad T(r_2, \theta) = T_0$.

The maximum deviation of the analytical solution of this equation from the numerical solution of equation (4) is less than 2% for $\epsilon = 0.1$ and $r_1/r_2 = 0.4$. Consequently, with a sufficiently high degree of accuracy, the temperature distribution for $\epsilon < 0.1$ can be considered as one-dimensional.

We compute the stress-strain state under conditions of thermoelastic equilibrium at $\epsilon = 0$. System of equilibrium equations (2) is transformed to

$$\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{2\sigma_{r\theta}}{r} = 0. \tag{6}$$

The components of the displacement vector are represented in the following form

$$u_r(r,\theta) = F(r) + R_0 C \sin(\theta), \quad u_\theta(r,\theta) = R_0 C \cos(\theta),$$
$$F(r) = \frac{\gamma}{r}\int_{r_1}^{r} \Delta(\rho)\rho d\rho + Ar + \frac{B}{r}, \quad \Delta(r) = \alpha r(T(r) - T_0), \quad \gamma = \frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)}. \tag{7}$$

where A, B, C is unknown constant. The stress tensor components are obtained from (3) according to relations (7). Constants A, B can be found from free thermal expansion condition (5). Constant C is calculated from equation

$$\int_{r_1}^{r_2} 2\pi\sigma_{\varphi\varphi}(\rho)\rho d\rho = 0 \tag{8}$$

PLASTIC FLOW

The process of thermoelastic deformation calculating with increasing temperature T_k is limited by the probability of the stressed state reaching the yield surface (Tresca prism). A plastic flow arises on the inner surface of the torus under the Tresca condition as follows

$$\sigma_{rr} - \sigma_{\theta\theta} = 2k, \quad \sigma_{rr} - \sigma_{\varphi\varphi} = 2k, \tag{9}$$

wherein $k = k_0(1 - \beta\Delta)$ denotes yield strength, β is the rate of yield strength dependence on the temperature increasing. The boundary b of the plasticity domain moves in the direction of the outer surface with temperature T_k increasing. Functions (7) of displacements and stresses in thermoelastic domain ($b < r < r_2$) remain valid up to new integration constants A, B, C.

Plastic domain consists of the two parts: complete plasticity domain ($r_1 < r < a$) corresponding to the Tresca prism edge under condition (9), plasticity domain ($a < r < b$) corresponding to the Tresca prism facet under condition $\sigma_{rr} - \sigma_{\varphi\varphi} = 2k$.

The problem is statically determinate in domain ($r_1 < r < a$) then stresses can be found as solution of the equilibrium system (2) under conditions (5) and (9)

$$\sigma^*_{rr} = -\frac{2}{r}\int_{r_1}^{r} \frac{k(\rho)}{\rho} d\rho, \quad \sigma^*_{\theta\theta} = \sigma^*_{\varphi\varphi} - \frac{2}{r}\int_{r_1}^{r} \frac{k(\rho)}{\rho} d\rho - 2k. \tag{10}$$

The displacements in this domain can again be represented as a sum of functions

$$u_r(r,\theta) = F^*(r) + R_0 C \sin(\theta), \quad u_\theta(r,\theta) = R_0 C \cos(\theta),$$
$$F^*(r) = \frac{3}{r}\int_{r_1}^{r} \Delta(\rho)\rho d\rho - \frac{1}{(3\lambda + 2\mu)}\left(r\int_{r_1}^{r} \frac{k(\rho)}{\rho} d\rho + \frac{1}{r}\int_{r_1}^{r} k(\rho)\rho d\rho\right) + Cr + \frac{D}{r}, \tag{11}$$

where D is the integration constant.

The displacements in the domain $a < r < b$ are represented in the form

$$u^{**}_r(r,\theta) = F^{**}(r) + R_0 C \sin(\theta), \quad u^{**}_\theta(r,\theta) = R_0 C \cos(\theta),$$

$$F^{**}(r) = \frac{\psi}{2\eta}\left(\frac{(\eta+1)}{r^\eta}\int_{r_1}^{r} \Delta(\rho)\rho^\eta d\rho + (\eta - 1)r^\eta \int_{r_1}^{r} \frac{\Delta(\rho)}{\rho^\eta} d\rho\right) + rC -$$
$$- \frac{1}{2(\lambda + \mu)}\left(\frac{1}{r^\eta}\int_{r_1}^{r} k(\rho)\rho^\eta d\rho + r^\eta \int_{r_1}^{r} \frac{k(\rho)}{\rho^\eta} d\rho\right) + Mr^\eta + \frac{N}{r^\eta}, \tag{12}$$

where M, N are the integration constants. The system of equation for the constants A, B, C, D, M, N consists of the conditions (5) and (8) and continuity conditions of radial stresses and displacements at the plastic boundaries a and b.

CONCLUSION

Solutions for stresses and displacements obtained under conditions of thermoelastic equilibrium were compared with numerical results for various values of the parameter ϵ. It was found that for $\epsilon < 0.1$ the maximum deviation of analytical solutions for stresses is less than 4% of numerical calculations. In this case, the solutions for displacements differ by less than 1% of the numerical counterparts. Thus, it can be concluded that the obtained analytical solution can be used with a high degree of accuracy to calculate the stress-strain state of thermoelastic material at positive values of the parameter ϵ. It is obvious that these solutions are useful in modelling the plastic flow process, since they allow one to obtain the most simple way of stresses and displacements calculations in the plastic flow domains and ensure continuity of the functions at elastic-plastic boundaries. Further possible investigation of the stress-strain state under toroidal symmetry is associated with the construction of approximate solutions taking into account the non-stationarity of the temperature gradient and the possibility of the appearance of a repeated plastic flow during unloading processes.

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