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PARTIAL DIFFERENTIAL EQUATIONS

A Singular Elliptic Boundary Value Problem in Domains with Corners: II. The Boundary Value Problem

V. V. Katrakhov and S. V. Kiselevskaya

Institute of Applied Mathematics, Far East Branch of the Russian Academy of Sciences, Vladivostok, Russia
Vladivostok State University of Service Economics, Vladivostok, Russia

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The present paper is a continuation of [1] and uses the notation and definitions introduced there. In [1], the authors also presented the properties of transformation operators, studied new function spaces, introduced the notion of σ-trace, and proved the direct and inverse trace theorems. In the present paper, we pose a singular boundary value problem in a domain with a single corner (singular) point. We prove some auxiliary results. The main result is given by the theorem on the unique solvability of the corresponding boundary value problem.

1. MAIN RESULT

In the Euclidean two-dimensional space $\mathbb{R}^2$, we introduce polar coordinates $r > 0$, $0 \le \varphi < 2\pi$ centered at the origin O. By S_R we denote a circular sector of radius R centered at O with opening angle $\Phi \in (0, 2\pi]$.

Consider a bounded domain $\Omega \subset \mathbb{R}^2$. Suppose that the origin O belongs to Ω. We assume that for some $R_0 > 0$, the intersection of Ω with the disk of radius $2R_0$ centered at O coincides with S_{2R_0}. Furthermore, we assume that the boundary of Ω is C^∞ everywhere except for the point O, which is assumed to be a corner point. Let $G_O = \partial\Omega \backslash O$.

Consider the boundary value problem

$$\Delta u = f(x), \qquad x \in \Omega, \tag{1}$$

$$u|_{\Gamma_O} = 0, \qquad x \in G_O, \tag{2}$$

$$\sigma u|_O = \Psi(\varphi), \qquad \varphi \in [0, \Phi]. \tag{3}$$

This problem generates the operator

$$\mathscr{A} : \ u \mapsto \Lambda u \equiv \{\Delta u, \sigma u|_O\}.$$

We equip the space $\mathscr{M}^s = M^s \times A[0, \Phi]$ with the direct product topology. It follows from the preceding results that the operator Λ is a continuous mapping of the space M^{s+2} into the space $\mathscr{M}^s$, where $s \ge 0$ is even.

The following assertion is the main result of the present paper.

Theorem 1. *Let $s \ge 0$ be even, and let $f \in M^s(\Omega)$ and $\Psi \in A[0, \Phi]$. Then there exists a unique solution u of the above-posed boundary value problem in the space $M^{s+2}(\Omega)$. Moreover, $f : \Psi \to u$ is a continuous mapping of the space $\mathscr{M}^s$ into the space $M^{s+2}(\Omega)$.*

Proof. The proof of the theorem consists of several stages. First, let us prove the following assertion.

2. AUXILIARY RESULTS

Theorem 2 (the uniqueness theorem). *Let $s \geq 0$. Then the homogeneous boundary value problem* (1)–(3) *has at most one solution in the space $M^{s+2}(\Omega)$.*

Proof. Let a solution u of the homogeneous boundary value problem belong to the space $M^{s+2}(\Omega)$. The harmonic function u admits the expansion

$$u(r,\varphi) = \sum_{k=1}^{\infty} \left(a_k r^{\lambda_k} + b_k r^{-\lambda_k}\right) Y_k(\varphi) \equiv u' + u''$$

for each $r \in (0, 2R_0)$. Since the σ-trace of u is equal to the σ-trace of u'', it follows that both of them vanish, i.e.,

$$0 = \sigma u''|_O = \sum_k b_k Y_k;$$

therefore, all b_k are zero, and so $u'' = 0$. Hence it follows that, in this case, the harmonic function u equal to u' belongs to the class $H^s_\Delta(\Omega)$ and, in particular, to the class $H^1_\Delta(\Omega) = H^1(\Omega)$. Since the homogeneous boundary value problem (1), (2) has the unique trivial solution in the latter space [2, 3], the proof of the theorem is complete.

Let us now proceed to the proof of the existence of a solution of the inhomogeneous boundary value problem (1)–(3). In forthcoming considerations, we need the following assertion.

Lemma 1. *Let $s \geq 0$ be even, and let a function $f \in M^s(S_R)$ vanish near the circular part of the boundary of the sector S_R. Then there exists a function $u \in M^{s+2}(S_R)$ satisfying the Poisson equation*

$$\Delta u = f(x) \tag{4}$$

and the homogeneous boundary condition

$$\sigma u|_O = 0 \tag{5}$$

at the corner point and such that $f \mapsto u$ is a continuous mapping of the space $M^s(S_R)$ into the space $M^{s+2}(S_R)$.

Proof. Let $f \in \mathring{T}^\infty(S_R)$. This implies the expansion $f(r,\varphi) = \sum_{k=0}^{K} f_k(r) Y_k$, where $K = K(f)$ is a positive integer and the functions $r^{-\lambda_k} f_k$ belong to the space $\mathring{C}^\infty_\nu(0,R)$. The desired solution u has the form

$$u(r,\varphi) = -\sum_{k=0}^{K} Y_k r^{\lambda_k} \int_r^{\bar{R}} t^{1-2\lambda_k} \int_0^t \tau^{\lambda_k+1} f_k(\tau)\, d\tau\, dt. \tag{6}$$

By Lemma 1 in [1], the functions in the space $\mathring{C}^\infty_\nu(0,R)$ have at most a power-law singularity of order -2ν at zero. Hence it follows that the integral is $O\left(r^{2-2\nu}\right)$ near $r = 0$; consequently, condition (5) is satisfied. The verification of condition (4) can be performed by straightforward differentiation in (6). Let us show that the mapping $f \mapsto u$ given by (6) is continuous in the corresponding topologies.

By u_k we denote the function $u_k = -r^{\lambda_k} \int_r^{\bar{R}} t^{-1-2\lambda_k} \int_0^t \tau^{\lambda_k+1} f_k(\tau)\, d\tau\, dt$. We expand it into two terms,

$$\begin{aligned} u_k = {} & -r^{\lambda_k} \int_r^{\bar{R}} t^{-1-2\lambda_k} \int_0^t \tau^{\lambda_k+1} \chi_{R/4} f_k(\tau)\, d\tau\, dt \\ & - r^{\lambda_k} \int_r^{\bar{R}} t^{-1-2\lambda_k} \int_0^t \tau^{\lambda_k+1} \left(1 - \chi_{R/4}\right) f_k(\tau)\, d\tau\, dt \stackrel{\text{def}}{=} u_k^1 + u_k^2, \end{aligned}$$

introduce the functions $u^1 = \sum_{k=0}^{K} u_k^1 Y_k$ and $u^2 = \sum_{k=0}^{K} u_k^2 Y_k$, and separately estimate each of them, starting from u^1. Consider the expression

$$B_\nu\left(\chi_R r^{-\lambda_k} u_k^1\right) = D^2\left(\chi_R r^{-\lambda_k} u_k^1\right) + \frac{2\nu+1}{r} D\left(\chi_R r^{-\lambda_k} u_k^1\right).$$

Here and throughout the following, ν is understood as λ_k. By the Leibniz formula, we obtain

$$\begin{aligned} B_\nu\left(\chi_R r^{-\lambda_k} u_k^1\right) &= D^2\chi_R r^{-\lambda_k} u_k^1 + D\chi_R D\left(r^{-\lambda_k} u_k^1\right) + D\chi_R D\left(r^{-\lambda_k} u_k^1\right) \\ &\quad + \chi_R D^2\left(r^{-\lambda_k} u_k^1\right) + \frac{2\nu+1}{r}\left(D\chi_R r^{-\lambda_k} u_k^1 + \chi_R D\left(r^{-\lambda_k} u_k^1\right)\right) \\ &= \chi_R B_\nu\left(r^{-\lambda_k} u_k^1\right) + 2\frac{\partial \chi_R}{\partial r}\frac{\partial\left(r^{-\lambda_k} u_k^1\right)}{\partial r} + r^{-\lambda_k} u_k^1 B_\nu \chi_R. \end{aligned} \tag{7}$$

For the first term in the last relation, we have the formula

$$\chi_R B_\nu\left(r^{-\lambda_k} u_k^1\right) = \chi_R r^{-\lambda_k}\chi_{R/4} f_k(r) = \chi_{R/4} r^{-\lambda_k} f_k(r),$$

since $\chi_R\chi_{R/4} = \chi_{R/4}$.

By taking into account the relations $D\chi_R(r) = 0$ for $0 \le r \le R$ and $\chi_{R/4}(r) = 0$ for $r \ge R/2$, for the second term, we obtain the expression

$$2\frac{\partial \chi_R}{\partial r}\frac{\partial\left(r^{-\lambda_k} u_k^1\right)}{\partial r} = 2r^{-1-2\lambda_k}\frac{\partial \chi_R}{\partial r}\int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau. \tag{8}$$

In the same way, for the third term, we obtain the representation

$$\begin{aligned} r^{-\lambda_k} u_k^1 B_\nu \chi_R &= -\left(B_\nu \chi_R\right)\int_r^{\bar{R}} t^{-1-2\lambda_k}\, dt \int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau \\ &= \left(B_\nu \chi_R\right)\frac{1}{2\lambda_k}\left(\bar{R}^{-2\lambda_k} - r^{-2\lambda_k}\right)\int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau. \end{aligned} \tag{9}$$

By substituting all these representations into (7), we obtain

$$\begin{aligned} B_\nu\left(\chi_R r^{-\lambda_k} u_k^1\right) &= \chi_{R/4} r^{-\lambda_k} f_k + \frac{2}{r}\frac{\partial \chi_R}{\partial r}\int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau \\ &\quad - \left(B_\nu \chi_R\right)\left(\bar{R}^{-2\lambda_k} - r^{-2\lambda_k}\right)\int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau \\ &= \chi_{R/4} r^{-\lambda_k} f_k + \int_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k\, d\tau\left(\frac{2}{r}\frac{\partial \chi_R}{\partial r} - \frac{1}{2\lambda_k} B_\nu \chi_R\left(\bar{R}^{-2\lambda_k} - r^{-2\lambda_k}\right)\right). \end{aligned} \tag{10}$$

By taking into account the relation

$$\frac{2}{r}\frac{\partial \chi_R}{\partial r} - \frac{1}{2\nu} B_\nu \chi_R = -\frac{1}{2\nu} B_{-\nu}\chi_R,$$

we rewrite the expression (10) in the form

$$B_\nu\left(\chi_R r^{-\lambda_k} u_k^1\right) = \chi_{R/4} r^{-\lambda_k} f_k - \frac{1}{2\lambda_k} B_{-\nu}\chi_R\left(\bar{R}^{-2\lambda_k} - r^{-2\lambda_k}\right)\int\limits_0^{R/2} \tau^{\lambda_k+1}\chi_{R/4} f_k d\tau.$$

By virtue of the formula $\|f\|_{s,R}^2 = \sum_{k=0}^K \left\|r^{-\lambda_k}\chi_R f_k\right\|^2_{\mathring{H}_\nu^s(0,2R)} + \|(1-\chi_R)f\|^2_{H_\Delta^s(\Omega)}$, one can estimate the function u^1 as

$$\begin{aligned}
\left\|u^1\right\|_{s+2,R}^2 &= \sum_k \left\|r^{-\lambda_k}\chi_R u_k^1\right\|^2_{\mathring{H}_\nu^{s+2}(0,2R)} + \left\|(1-\chi_R)u_k^1\right\|^2_{H_\Delta^{s+2}(\Omega)} \\
&\le 3\sum_k \left\|\chi_{R/4} r^{-\lambda_k} f_k\right\|^2_{\mathring{H}_\nu^s(0,2R)} \\
&\quad + 3\sum_k \frac{1}{(2\lambda_k)^2}\Bigg(\bar{R}^{-2\lambda_k}\|B_{-\nu}\chi_R\|^2_{\mathring{H}_\nu^s(0,2R)} \int\limits_0^{R/2}\tau^{\lambda_k+1}\chi_{R/4} f_k d\tau \\
&\quad + \left\|r^{-2\lambda_k} B_{-\nu}\chi_R\right\|^2_{\mathring{H}_\nu^s(0,2R)} \int\limits_0^{R/2}\tau^{\lambda_k+1}\chi_{R/4} f_k d\tau\Bigg) + \left\|(1-\chi_R)u_k^1\right\|^2_{H_\Delta^s(S_{\bar{R}})} \\
&\stackrel{\text{def}}{=} 3J_1 + 3J_2 + \left\|(1-\chi_R)u_k^1\right\|^2_{H_\Delta^s(S_{\bar{R}})}.
\end{aligned} \qquad (11)$$

Let us estimate each term on the right-hand side in the last formula. The term J_1 admits an estimate of the form $J_1 \le \|f\|_{s,R/4}^2$. We set $\omega(r) = r^{-\lambda_k}\chi_{R/4} f_k$ and consider the integral in the second term:

$$W_\nu(\omega, R) = \int\limits_0^{R/2} \tau^{2\lambda_k+1}\omega(r)d\tau = \int\limits_0^{R/2} \tau^{2\lambda_k+1}\mathscr{P}_\nu^{1/2-\nu}\mathscr{S}_\nu^{\nu-1/2}\omega(r)d\tau,$$

where $\mathscr{P}_\nu^{1/2-\nu}$ and $\mathscr{S}_\nu^{\nu-1/2}$ are transformation operators.

Since $r^{-\lambda_k} f_k \in \mathring{C}_\nu^\infty(0,R)$, it follows that the functions $\mathscr{S}_\nu^{\nu-1/2}\omega = \mathscr{S}_\nu^{\nu-1/2}\left(\chi_R r^{-\lambda_k} f_k\right)$ belong to the space $\mathring{C}^\infty[0,R)$. Let $\tilde{\omega} = \mathscr{S}_\nu^{\nu-1/2}\omega$ and $\nu < N + 1/2$, where N is a positive integer. Then, by the definition of the transformation operators, we have

$$P_\nu^{1/2-\nu}\tilde{\omega}(\tau) = \frac{(-1)^N \times 2^{-N}\sqrt{\pi}\tau^{2(N-\lambda_k)}}{\Gamma(\nu+1)\Gamma(N-\nu+1/2)}\left(\frac{\partial}{\partial\tau}\frac{1}{\tau}\right)^N \int\limits_\tau^\infty \tau^{2\lambda_k}\left(t^2-\tau^2\right)^{N-\lambda_k-1/2} t^{-2N}\tilde{\omega}(t)dt.$$

Hence we obtain an expression of the form

$$\begin{aligned}
W_\nu(\omega, R) &= \int\limits_0^{R/2} \tau^{2\lambda_k+1} P_\nu^{1/2-\nu} S_\nu^{\nu-1/2}\omega\, d\tau \\
&= \frac{(-1)^N \times 2^{-N}\sqrt{\pi}}{\Gamma(\nu+1)\Gamma(N-\nu+1/2)} \int\limits_0^{R/2} \tau^{2\lambda_k+1}\left(\frac{\partial}{\partial\tau}\frac{1}{\tau}\right)^N \\
&\quad \times \int\limits_\tau^\infty \tau^{2\lambda_k}\left(t^2-\tau^2\right)^{N-\lambda_k-1/2} t^{-2N}\tilde{\omega}(t)dt\, d\tau \\
&= \frac{\Gamma(N+3/2)\sqrt{\pi}}{\Gamma(\nu+1)\Gamma(N-\nu+1/2)\Gamma(3/2)} \int\limits_0^{R/2} \tilde{\omega}(t)t^{-2N} \int\limits_0^t \tau^{2\lambda_k+1}\left(t^2-\tau^2\right)^{N-\lambda_k-1/2} dt\, d\tau.
\end{aligned}$$

The inner integral in the last expression can be evaluated via the Euler functions:

$$\int_0^t \tau^{2\lambda_k+1}\left(t^2-\tau^2\right)^{N-\lambda_k-1/2} d\tau = t^{2N+1}\frac{\Gamma(\nu+1)\Gamma\left(N-\lambda_k+1/2\right)}{2\Gamma(N+3/2)};$$

consequently,

$$W_\nu(\omega,R)=\int_0^{R/2} t\tilde{\omega}(t)dt=\int_0^{R/2} tS_\nu^{\nu-1/2}\omega\,dt.$$

Further, since $S_\nu = I^{1/2-\nu}S_\nu^{\nu-1/2}$, where I^μ is the Liouville operator, it follows from the preceding formula that

$$W_\nu=\int_0^{R/2} tI^{s+\nu-1/2}I^{-s}S_\nu\omega(t)dt=\frac{1}{\Gamma(s+\nu-1/2)}\int_0^{R/2} t\int_t^{R/2}(\tau-t)^{s+\lambda_k-3/2}I^{-s}S_\nu\omega(\tau)d\tau\,dt$$

$$=\frac{1}{\Gamma(s+\nu-1/2)}\int_0^{R/2}\left(I^{-s}S_\nu\omega(\tau)\right)\int_0^{\tau} t(\tau-t)^{s+\lambda_k-3/2}dt\,d\tau,$$

and since

$$\int_0^{\tau} t(\tau-t)^{s+\lambda_k-3/2}dt=\tau^{s+\lambda_k+1/2}\frac{\Gamma(s+\nu-1/2)}{\Gamma(s+\nu+3/2)},$$

we have

$$W_\nu=\frac{1}{\Gamma(s+\nu+3/2)}\int_0^{R/2}\tau^{s+\lambda_k+1/2}I^{-s}S_\nu\omega(\tau)d\tau.$$

By the Cauchy–Schwarz inequality, we obtain an estimate of the form

$$|W_\nu|\le\frac{1}{\Gamma(s+\nu+3/2)}\left(\int_0^{R/2}t^{2s+2\lambda_k+1}dt\right)^{1/2}\left\|D^sS_\nu\omega\right\|_{L_2(0,R/2)}$$

$$=\frac{R^{s+\nu+1}}{2^{s+\lambda_k+3/2}(s+\nu+1)^{1/2}\Gamma(s+\nu+3/2)}\left\|S_\nu\omega\right\|_{\mathring{H}_\nu^s(0,R/2)}$$

$$=\frac{R^{s+\nu+1}\Gamma(\nu+1)}{2^{s+1}(s+\nu+1)^{1/2}\Gamma(s+\nu+3/2)}\|\omega\|_{\mathring{H}_\nu^s(0,R/2)};$$

i.e., we have

$$|W_\nu|\le c(s,R)R^\nu(\nu+1)^{-1-s}\|\omega\|_{\mathring{H}_\nu^s(0,R/2)}.$$

By returning to the original notation, we write out the final estimate of the integral W_ν :

$$|W_\nu(\omega,R)|\le c(s,R)R^{\lambda_k}\left(\lambda_k+1\right)^{-1-s}\left\|\chi_{R/4}r^{-\lambda_k}f_k\right\|_{\mathring{H}_\nu^s(0,R/2)}. \tag{12}$$

It follows from [4, p. 854] that

$$\|B_\nu\chi_R\|^2_{\mathring{H}_\nu^s(0,2R)}\le 2\left\|B_\nu\chi_R\right\|^2_{\mathring{H}_{\nu,+}^s(0,2R)}=2\int_0^{2R}\left|B_\nu^{1+s/2}\chi_R\right|^2 r^{2\lambda_k+1}dr$$

$$=2R^{2\lambda_k+1-s}\int_0^{2R}\left|B_\nu^{1+s/2}\chi_R\right|^2t^{2\lambda_k+1}dt=c(s,k)\left(\lambda_k+1\right)^{s+1}(2R)^{2\lambda_k},$$

and we have the inequality

$$\left\| r^{-2\nu} B_\nu \chi_R \right\|^2_{\mathring{H}^s_\nu(0,2R)} \le c(s,R) R^{-2\lambda_k} \left(\lambda_k + 1\right)^{s+1}.$$

Therefore, by (12), the last two inequalities lead to the following estimate for the second term in (11):

$$I_2 \le c \sum_k \left(\frac{R^{2\lambda_k}}{\left(\lambda_k+1\right)^{4+2s}} \left(\frac{\lambda_k+1}{R^{2\lambda_k}} + \frac{(2R)^{2\lambda_k}}{\bar{R}^{4\lambda_k}} \left(\lambda_k+1\right)^{s+1} \right) \left\| \frac{\chi_{R/4}}{r^{\lambda_k}} f_k \right\|^2_{\mathring{H}^s_\nu(0,R/2)} \right)$$
$$\le c \sum_k \left\| \frac{\chi_{R/4}}{r^{\lambda_k}} f_k \right\|^2_{\mathring{H}^s_\nu(0,R/2)} \le c \|f\|^2_{s,R/4},$$

since $2R < \bar{R}$. Here $c > 0$ is independent of f.

To complete the estimate of the function u^1, it remains to consider the last term in (11). Since $\chi_{\bar{R}}(r) \equiv 1$ in $S_{\bar{R}}$, we have

$$\left\| \left(1-\chi_R\right) u^1 \right\|_{H(S_{\bar{R}})} \le \left\| \chi_R \left(1-\chi_R\right) u^1 \right\|_{H(S_{2\bar{R}})}.$$

By analogy with (7)–(10), we obtain the formula

$$B_\nu \left(\chi_R \left(1-\chi_R\right) r^{-\lambda_k} u^1_k \right) = \frac{1}{2\lambda_k} \left(\frac{B_\nu \left(\chi_R \left(1-\chi_R\right)\right)}{\bar{R}^{2\lambda_k}} - \frac{B_{-\nu} \left(\chi_R \left(1-\chi_R\right)\right)}{r^{2\lambda_k}} \right) \int_0^{R/2} \tau^{\lambda_k+1} \chi_{R/4} f_k \, d\tau.$$

Consequently,

$$\left\| \chi_{\bar{R}} \left(1-\chi_R\right) u^1 \right\|^2_{\mathring{H}^{s+2}(S_{2\bar{R}})}$$
$$\le \sum_k \frac{|W_\nu|^2}{(2\nu)^2} \left(\left\| \frac{B_\nu^{1+s/2} \left(\chi_{\bar{R}} \left(1-\chi_R\right)\right)}{\bar{R}^{4\nu}} \right\|_{L_{2,\nu}} + \left\| \frac{B_{-\nu}^{1+s/2} \left(\chi_{\bar{R}} \left(1-\chi_R\right)\right)}{r^{2\nu}} \right\|_{L_{2,\nu}} \right).$$

The norms in the last sum satisfy the estimate

$$\left\| \chi_R \left(1-\chi_R\right) u^1 \right\|^2_{\mathring{H}^{s+2}_\Delta(S_{\bar{R}})} \le c \sum_k \frac{1}{\left(\lambda_k+1\right)^{s+1}} \left\| \frac{\chi_{R/4}}{r^{\lambda_k}} f_k \right\|^2_{\mathring{H}_\nu(0,R/2)} \le c \|f\|^2_{s,R/4};$$

therefore,

$$\left\| \left(1-\chi_R\right) u^1 \right\|_{H(S_{\bar{R}})} \le c \|f\|_{s,R/4}.$$

We have thereby estimated the function u^1 as $\|u^1\|_{s,R/4} \le c\|f\|_{s,R/4}$, where $c > 0$ is a constant independent of f.

Now consider the function u^2. It belongs to the space $H^{s+2}_\Delta\left(S_{\bar{R}}\right)$ and is a solution of the boundary value problem

$$\Delta u^2 = \left(1-\chi_{R/4}\right) f, \qquad u^2\big|_{\partial(S_{\bar{R}})} = 0.$$

Since $f \in M^s\left(S_{\bar{R}}\right)$, it follows that $\left(1-\chi_{R/4}\right)$ belongs to $H^s_\Delta\left(S_{\bar{R}}\right)$. The solution of this boundary value problem is unique and admits the estimate (see the proof of the main theorem below)

$$\left\| u^2 \right\|_{H^{s+2}_\Delta(S_R)} \le c \left\| \left(1-\chi_{R/4}\right) f \right\|_{H^s_\Delta(S_R)}.$$

Now from the definition of the norms $\|\cdot\|_{s,R}$, we have $\|u^2\|_{s+2,R} \le c\|f\|_{s,R}$, where c is a constant independent of f.

We have thereby obtained estimates for the functions u^1 and u^2, i.e., for the function $u = u^1 + u^2$ as well. Consequently, for any $R \in \left(0, \bar{R}/2\right)$ and $s \geq 0$, there exists a constant $c > 0$ such that $\|u\|_{s+2,R} \leq c\|f\|_{s,R/4}$ for any function $f \in T^\infty\left(S_{\bar{R}}\right)$.

To complete the proof, we perform the passage to the limit. Let $f \in M^s\left(S_{\bar{R}}\right)$, and let this function satisfy the assumptions of the lemma. Then there exists a function sequence $f^m \in T^\infty\left(S_{\bar{R}}\right)$ converging to f in the topology of this space. For each function f^m, we define the functions u^m by formula (6). Then

$$\Delta u^m = f^m \overset{m \to \infty}{\to} f. \tag{13}$$

As was proved above, we have $\|u^m\|_{s+2,R} \leq c\,\|f^m\|_{s,R}$; therefore, $f^m \mapsto u^m$ is a continuous mapping of the space M^s into M^{s+2}. Then u^m is a Cauchy sequence in M^{s+2}. Since the space M^{s+2} is complete, it follows that there exists a function $u \in M^{s+2}$ that is the limit of the sequence u^m in the topology of this space.

The operator Δ is a continuous mapping of the space M^{s+2} into M^s. Since the space M^{s+2} is continuously embedded in M^s, we have $\|g\|_{s,R} \leq c\|g\|_{s+2,R}$ for any function $g \in M^s$. In particular, for the function g, one can take $\Delta u^m \in M^s$; then

$$\|\Delta u^m\|_{s,R} \leq c\,\|\Delta u^m\|_{s+2,R}\,,$$

and this implies that the operator Δ is a continuous mapping of M^{s+2} into M^s. Therefore, $\Delta u^m \to \Delta u$ in the sense of the space M^s. Then relation (13) implies that $\Delta u = f$.

Finally, by the direct theorem on σ-traces, the passage from a function to its σ-trace is a continuous operation; therefore,

$$\lim_{m \to \infty} \sigma u^m\Big|_O = \sigma u|_O,$$

and since $\sigma u^m|_O = 0$, we have $\sigma u|_O = 0$ for the σ-trace. The proof of the lemma is complete.

3. EXISTENCE OF A SOLUTION OF THE BOUNDARY VALUE PROBLEM

Let us prove the solvability of problem (1)–(3). By G' we denote the part of the boundary of S_R that consists of two rectilinear segments, the corner sides. Let $v^1 \in M^{s+2}(\Omega)$ be the solution of the boundary value problem

$$\Delta v^1 = 0, \qquad x \in S_\infty, \qquad v^1\big|_{G'} = 0, \qquad \sigma\, v^1\big|_O = \Psi$$

represented in Theorem 6 in [1], and let $v^2 \in M^{s+2}(\Omega)$ be a solution of the boundary value problem

$$\Delta v^2 = \chi_{R_0} f, \qquad x \in S_{2\bar{R}}, \qquad v^2\big|_{G'} = 0, \qquad \sigma v^2\big|_O = 0$$

constructed in Lemma 1.

Now let us analyze the solvability of the boundary value problem

$$\Delta v^3 = \left(1 - \chi_{R_0}\right) f, \qquad x \in \Omega, \qquad v^3\big|_G = -\left(v^1 + v^2\right)\big|_G\,, \qquad \sigma v^3\big|_O = 0. \tag{14}$$

Note that, as was proved above, the function $v^1 + v^2$ belongs to the space $M^{s+2}(\Omega)$, whose elements have the same structure as the functions of the class H^s at some distance from the origin; therefore, the trace $\left(v^1 + v^2\right)\big|_G$ exists and belongs to the space $H^{s+3/2}(G)$. Here we also use the fact that this trace vanishes on the part of the boundary G lying in some neighborhood of the origin. This fact, together with the general theory of elliptic boundary value problems (e.g., see [2, 3]), permits one to prove the existence of a solution of the boundary value problem (14) in the class $H^1(\Omega) = H^1_\Delta(\Omega)$, which, by smoothness increasing theorems (e.g., see [5]), locally (at some distance from the origin) belongs to the class H^{s+2}, but, for some values of angles, it does not necessarily belong even to the space $H^2(\Omega)$. In addition, this solution belongs to the space $H^{s+2}_\Delta(\Omega)$, since

$$\Delta^{(s+2)/2} v^3 = \Delta^{s/2}\left(1 - \chi_{R_0}\right) f,$$

i.e., $v^3 \in H^{s+2}_\Delta(\Omega) \subset M^{s+2}(\Omega)$.

Then the function $v = v^1 + v^2 + v^3 \in M^{s+2}(\Omega)$ is a solution of the boundary value problem (1)–(3). The proof of the existence of a solution is thereby complete.

The uniqueness of the solution of the boundary value problem (1)–(3) was justified in Theorem 1. The preceding results also imply the continuity of the resolving operator $\mathscr{A}$. The proof of Theorem 2 is complete.

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